KEY

TO

ATKINSON'S BRIDGE'S

ALGEBRA.

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CONTENTS.

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ATKINSON'S BRIDGE'S ALGEBRA.

CHAPTER I.—Definitions.

(p. 3.)—Ex. 3. If $a=5$, $b=4$, $c=3$, $d=2$, $x=1$, $y=0$; find the numerical values of the following expressions:

(1.) $a+b+c+x=5+4+3+1$
$=13$

(2.) $a-b+c-x+y=5-4+3-1+0$
$=1+3-1$
$=4-1$
$=3.$

(3.) $ab+3ac-bc+4cx-xy=5\times4+3\times5\times3-4\times3$
$+4\times3\times1-1\times0=20+45-12+12-0$
$=77-12$
$=65.$

(4.) $abc-abd+bcd-acx=5\times4\times3-5\times4\times2+4\times3$
$\times2-5\times3\times1=60-40+24-15$
$=84-40-15$
$=29.$

(5.) $3abc+4acx-8bdx+axy=3\times5\times4\times3+4\times5\times3$
$\times1-8\times4\times2\times1+5\times1\times0=180+60-64+0$
$=240-64$
$=176.$

Ex. 2. If $a=3$, $b=2$, $c=1$; find the numerical values of

(1.) $\frac{3a+c}{4b+a}=\frac{3\times3+1}{4\times2+3}=\frac{9+1}{8+3}=\frac{10}{11}$

(2.) $\frac{a+2b-c}{3a+b-5c}=\frac{3+2\times2-1}{3\times3+2-5\times1}=\frac{3+4-1}{9+2-5}=\frac{6}{6}=1.$

(3.) $\frac{ab+ac-bc}{2ab-2ac+bc}=\frac{3\times2+3\times1-2\times1}{2\times3\times2-2\times3\times1+2\times1}$

$=\frac{6+3-2}{12-6+2}=\frac{7}{8}.$

(p. 5.)—Ex. 4. If $a=1$, $b=3$, $c=5$, $d=0$; find the values of

(1.) $a^2+2b-c=1^2+2\times3-5$
$=1+6-5$
$=2.$

(2.) $a^2+3b^2-c^2=1^2+3\times3^2-5^2$
$=1+3\times9-25$
$=1+27-25$
$=3.$

(3.) $a^2+2b^2+3c^2+4d^2=1^2+2\times3^2+3\times5^2+4\times0^2$
$=1+2\times9+3\times25+4\times0$
$=1+18+75+0$
$=94$

(4.) $3a^2b-2b^2c+4c^2-4a^2d=3\times1^2\times3-2\times3^2\times5$
$+4\times5^2-4\times1^2\times0=3\times1\times3-2\times9\times5+4\times25-4\times1\times0$
$=9-90+100-0$
$=19.$

(5.) $a^3+b^3=1^3+3^3$
$=1+27$
$=28.$

(6.) $\frac{a^3}{3}+\frac{b^3}{3}+\frac{c^3}{3}=\frac{1^3}{3}+\frac{3^3}{3}+\frac{5^3}{3}$

$=\frac{1}{3}+\frac{27}{3}+\frac{125}{3}$

$=\frac{153}{3}$

$=51.$

Ex. 5. Let $a=64$, $b=81$, $c=1$; find the values of

(1.) $\sqrt{a}+\sqrt{b}=\sqrt{64}+\sqrt{81}.$
$=8+9$
$=17.$

(p. 5.) (2.) $\sqrt{a}+\sqrt{b}+\sqrt{c}=\sqrt{64}+\sqrt{81}+\sqrt{1}.$

$$=8+9+1$$
$$=18.$$

(3.) $\sqrt{abc}=\sqrt{64\times 81\times 1}$

$$=\sqrt{5184}$$
$$=72.$$

CHAP. II.—Addition.

Case I. (p. 7.)

Ex. 4.

$$\begin{array}{r} 3x^3+4x^2-\ x \\ 2x^3+\ x^2-3x \\ 7x^3+2x^2-2x \\ 4x^3+\ x^2-\ x \\ \hline 16x^3+8x^2-7x \end{array}$$

Ex. 5.

$$\begin{array}{r} 7a^3-3a^2b+\ 2ab^2-\ 3b^3 \\ 4a^3-\ a^2b+\ ab^2-\ b^3 \\ a^3-2a^2b+\ 3ab^2-\ 5b^3 \\ 5a^3-3a^2b+\ 4ab^2-\ 2b^3 \\ \hline 17a^3-9a^2b+10ab^2-11b^3 \end{array}$$

Ex. 6.

$$\begin{array}{r} 2x^2y-\ 3x+\ 2 \\ 4x^2y-\ 2x+\ 1 \\ 3x^2y-\ 5x+10 \\ x^2y-\ x+15 \\ \hline 10x^2y-11x+28 \end{array}$$

Case II. (p. 8.)

Ex. 4.

$$\begin{array}{r} 4x^3-\ 2x+3y \\ -\ x^3+\ 4x-\ y \\ 7x^3-\ x+9y \\ 9x^3+21x-2y \\ \hline 19x^3+22x+9y \end{array}$$

Ex. 5.

$$\begin{array}{r} 5a^3-2ab+\ b^2 \\ -\ a^3+\ ab-2b^2 \\ 4a^3-3ab+\ b^2 \\ 2a^3+4ab-4b^2 \\ \hline 10a^3\ \ *\ -4b^2 \end{array}$$

Ex. 6.

$$\begin{array}{r} 4x^2y^2+2xy-3 \\ -\ x^2y^2-\ xy-1 \\ 3x^2y^2+4xy-5 \\ -9x^2y^2-2xy+9 \\ \hline -3x^2y^2+3xy\ \ * \end{array}$$

SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

(p. 11.)—Ex. 5. Given $10x=150$; find the value of x.

Dividing both sides of the equation by 10, we get

$$x=15.$$

Ex. 6. $3x+4x+7x=84.$

Collecting the terms, $14x=84,$

Dividing by 14, $x=6.$

Ex. 7.
$$8x-5x+4x-2x=25.$$
$$12x-7x=25,$$
$$5x=25;$$
$$\therefore x=5.$$

Ex. 8.
$$12x-3x-4x-x=24.$$
$$12x-8x=24,$$
$$4x=24;$$
$$\therefore x=6.$$

Problems.

(p. 15.)—Prob. 6. A horse and a saddle were bought for £40, but the horse cost 9 times as much as the saddle; what was the price of each?

Let x = the price of the saddle in £,
then $9x$ = ... horse ......

But price of the saddle + price of the horse = £40.

$\therefore x+9x=40,$
$10x=40;$
$\therefore x=$ £4, the price of the saddle,
and $9x=$ £36, ... horse.

Prob. 7. Divide two dozen marbles between Richard and Andrew, so that Richard may have three times as many as Andrew.

Let x = the number Andrew receives,
then $3x$ = ... Richard ....

But the No. to be divided between them is two doz., or 24,

$\therefore x+3x=24,$
$4x=24;$
$\therefore x=6$, Andrew's share,
and $3x=18$, Richard's ......

Prob. 8. A boy being asked how many marbles he had, said, If I had twice as many more, I should have 36. How many had he?

Let x = the number of marbles the boy had,
then $2x$ = twice as many as he had.

But the marbles the boy had, and twice as many besides, make 36,

$\therefore x+2x=36,$
$3x=36;$
$\therefore x=12$

(p. 17.)—Prob. 13. A gentleman meeting four poor persons, gave five shillings amongst them; to the second he gave twice, to the third thrice, and to the fourth four times as much as to the first. What did he give to each?

Let $x =$ the No. of pence he gave to the 1st person,
then $2x =$ .... 2d ....
$3x =$ .... 3d ....
and $4x =$ .... 4th ....

But the money which the gentleman gave among them was $5s.$, or $60d.$

$$\therefore x+2x+3x+4x=60,$$
$$10x=60;$$
$$\therefore x=6d.$$

Hence, he gave to each poor person respectively, $6d.$, $12d.$, $18d.$, and $24d.$

Prob. 14. Divide a line 12 feet long into three parts, such that the middle one shall be double the least, and the greatest triple the least part.

Let $x =$ length *in feet* of the least part,
then $2x =$ .... middle ......
and $3x =$ .... greatest ......

$$\therefore x+2x+3x=12,$$
$$6x=12;$$
$\therefore x=2$, length of the least part in feet;
$2x=4$, ... middle ....
$3x=6$, ... greatest ....

Prob. 15. Divide 40 into three such parts, that the first shall be 5 times the second, and the third equal to the difference between the first and second.

Let $x =$ the second part,
then $5x =$ the first ......
$\therefore 5x-x$ or $4x =$ the third ......

Now these three parts taken together are to make 40;

$$\therefore x+5x+4x=40,$$
$$10x=40;$$
$\therefore x=4$, the second part,
$5x=20$, the first ......
and $4x=16$, the third ......

Prob. 16. A grocer mixed three kinds of tea, Bohea at $3s.$ per ℔., Twankay at $5s.$, and Souchong at $7s.$ per ℔.

The mixture contains the same quantity of each, and cost £6. How many ℔s. are there of each kind?

Let $x=$ the number of ℔s. of each kind;
then $3x=$ the price of the Bohea *in shillings*,
$5x=$ .... Twankay ....
and $7x=$ .... Souchong ....

But the mixture, or the three kinds taken together, cost £6, or 120*s*.

$$\therefore\ 3x+5x+7x=120,$$
$$15x=120;$$
$$\therefore\ x=8, \text{ the No. of ℔s. of each kind.}$$

(p. 17.)—Prob. 17. A bill of £700 was paid in sovereigns, half-sovereigns, and crowns, and an equal number of each was used. Find the number.

Let x be the number of each species of coin;
then $20x=$ the value of the sovereigns *in shillings*,
$10x=$ .... half-sovereigns ....
$5x=$ .... crowns ....

Now, by the problem, the sum of their values is £700, or 14000*s*.

$$\therefore\ 20x+10x+5x=14000,$$
$$35x=14000;$$
$$\therefore\ x=400.$$

Prob. 18. Two travellers set out at the same time from Guildford and London, a distance of 27 miles apart; the one walks 4 miles an hour, and the other 5 miles. In how many hours will they meet?

Let x be the number of hours:
then $4x=$ number of miles the one walks,
and $5x=$ .... the other ......

But together they walk 27 miles;

$$\therefore\ 4x+5x=27,$$
$$9x=27;$$
$$\therefore\ x=3 \text{ hours.}$$

Prob. 19. A person bought a horse, chaise, and harness, for £120; the price of the horse was twice the price of the harness, and the price of the chaise twice the price of both horse and harness; what was the price of each?

Let $x=$ the price of the harness in £
then $2x=$ .... horse ......
and $6x=$ .... chaise ......

Therefore, by the problem,

$$x+2x+6x=120,$$
$$9x=120;$$
$$\therefore x=\pounds 13 \quad 6s.\ 8d., \text{ price of harness,}$$
$$2x=\pounds 26 \quad 13 \quad 4 \quad \ldots\ldots\ldots \text{ horse,}$$
$$\text{and } 6x=\pounds 80 \quad 0 \quad 0 \quad \ldots\ldots\ldots \text{ chaise.}$$

Subtraction.—(p. 18.)

	Ex. 6.	Ex. 7.	Ex. 8.
From	$7xy+2x-3y$	$14x+y-z-5$	$13x^3-2x^2+7$
Subtract	$2xy-x+y$	$x+y+z-11$	$-x^3+x^2-6$
Rem^r^.	$5xy+3x-4y$	$13x \quad * -2z+6$	$14x^3-3x^2+13$

SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

(p. 20.)—Ex. 4. $\quad 2x+3=x+17$

Here, by transposing the x to the *left-hand* side, and the 3 to the *right-hand* side of the equation, and changing their signs, we get

$$2x-x=17-3$$
$$\therefore x=14.$$

Ex. 5. $\quad 5x-4=4x+25$

By transposition, $5x-4x=25+4$;

$$\therefore x=29.$$

Ex. 6. $\quad 7x-9=6x-3$

$$7x-6x=9-3$$
$$\therefore x=6.$$

Ex. 7. $\quad 4x+2a=3x+9b$

Transposing, $4x-3x=9b-2a$

$$\therefore x \quad 9b \quad 2a.$$

(p. 20.) Ex. 8. $15x+4=34$

$$15x=34-4$$
$$15x=30;$$

$\therefore$ by dividing (RULE I.) both sides of the equation by 15,

$$x=2.$$

Ex. 9.

$$8x+7=6x+27$$
$$8x-6x=27-7$$
$$2x=20$$
$$\therefore\ x=10.$$

Ex. 10.

$$9x-3=4x+22.$$
$$9x-4x=22+3$$
$$5x=25$$

$\therefore$ Dividing both sides by 5, $x=5$.

Ex. 11. $17x-4x+9=3x+39$.

By transposition, $17x-4x-3x=39-9$

And collecting the terms, $10x=30$

Dividing (RULE I.) both sides of the equation by 10,

$$x=3.$$

Ex. 12. $ax-c=b+2c$.

By transposing c to the right-hand side, the equation becomes

$$ax=b+2c+c$$
$$\text{or } ax=b+3c$$

Divide (RULE I.) both sides of the equation by a, and then, since $\frac{ax}{a}=x$, we have

$$x=\frac{b+3c}{a}.$$

Ex. 15. $4x-(3x+4)=8$.

This equation becomes, after the bracket has been taken away,

$$4x-3x-4=8$$

By transposition, $4x-3x=8+4$;

$$\therefore x=12.$$

(p. 20.)—Ex. 16. $8x-(6x-8)=9-(3-x)$

Take away the brackets, then the equation becomes, after the signs of the quantities within them have been changed,

$$8x-6x+8=9-3+x.$$

And, transposing, $8x-6x-x=9-3-8$;

$$\therefore x=-2.$$

Ex. 17. $4x-(3x-6)-(4x-12)=12-(5x-10).$

$$4x-3x+6-4x+12=12-5x+10$$
$$4x-3x-4x+5x=12+10-6-12$$
$$2x=4;$$
$$\therefore x=2.$$

Ex. 18. $5x-(3+3x)=8-(-x-1).$

$$5x-3-3x=8+x+1$$
$$2x-3=9+x$$
$$2x-x=9+3;$$
$$\therefore x=12.$$

PROBLEMS.

(p. 21.)—Prob. 4. At an election 420 persons voted, and the successful candidate had a majority of 46. How many voted for each candidate?

Let $x=$ No. of votes the unsuccessful candidate had, then $x+46=$ .... the successful one had, and the number of persons who voted was 420;

$$\therefore x+x+46=420,$$
$$2x+46=420,$$
$$2x=420-46,$$
$$2x=374;$$

$\therefore x=187$, the No. of votes one had,

and $x+46=233$, ... the other had.

(p. 22.)—Prob. 5. One of two rods is 8 feet longer than the other, and the longer rod is three times the length of the shorter. What are their lengths?

Let $x=$ the length *in feet* of the shorter rod, then $x+8=$ .... longer ......
and $3x=3$ times the length in feet of the shorter.

But the longer rod is equal to 3 times the length of the shorter;

$$\therefore x+8=3x;$$

By transposition, $3x-x=8$,

$$2x=8;$$

$$\therefore x=4 \text{ feet, length of the shorter rod,}$$

and $x+8=12$ .... longer ......

(p. 22.)—Prob. 6. Five times a number diminished by 16 is equal to three times the number. What is the number?

Let x be the number,
then $3x=3$ times the number,
and $5x=5$ times ....

But when 16 has been subtracted from 5 times the number, the remainder is equal to 3 times the number;

$$\therefore 5x-16=3x,$$

$$5x-3x=16,$$

$$2x=16;$$

$$\therefore x=8, \text{ the number required.}$$

Prob. 8. A draper has three pieces of cloth, which together measured 159 yards; the second piece was 15 yards longer than the first, and the third 24 yards longer than the second. What is the length of each piece?

Let $x=$ the length in yards of the 1st piece,
then $x+15=$ .... 2d ......
and $x+15+24=$ .... 3d ......

But the length of the three pieces taken together is 159 yds.

$$\therefore x+(x+15)+(x+15+24)=159.$$

Collecting the terms, $3x+54=159$,

$$3x=159-54,$$

$$3x=105;$$

$\therefore x=35$ yds., length of 1st piece,
$x+15=50$ .... 2d ......
and $x+15+24=74$ .... 3d ......

Prob. 9. Divide £36 among three persons, A, B, and C, in such a manner that B shall have £4 more than A, and C £7 more than B.

Let $x=$ A's share in £
then $x+4=$ B's ....
and $x+11=$ C's ....

And the sum of their shares is £36;

$$\therefore x+x+4+x+11=36,$$
$$3x+15=36,$$
$$3x=36-15,$$
$$3x=21;$$
$$\therefore x=£7, \text{ A's share,}$$
$$x+4=£11, \text{ B's ......}$$
$$\text{and } x+11=£18, \text{ C's ......}$$

(p. 22.)—Prob. 10. A gentleman buys four horses; for the second he gives £12 more than for the first; for the third £5 more than for the second; and for the fourth £2 more than for the third. The sum paid for all the horses was £240. Find the price of each.

Let $x=$ the price in £ of the 1st horse,
then $x+12=$ .... 2d ......
$x+17=$ .... 3d ......
and $x+19=$ .... 4th ......

Now the sum paid for all the horses was £240,

$$\therefore x+x+12+x+17+x+19=240,$$
$$4x+48=240,$$
$$4x=240-48,$$
$$4x=192;$$
$$\therefore x=48.$$

The prices therefore of the four horses are £48, £60, £65, and £67, respectively.

(p. 24.)—Prob. 14. It is required to divide the number 99 into five such parts, that the first may exceed the second by 3, be less than the third by 10, greater than the fourth by 9, and less than the fifth by 16.

Assume the 1st part $=x+3$,
then ... 2d ... $=x$,
... 3d ... $=x+13$,
... 4th ... $=x-6$,
and ... 5th ... $=x+19$.

And the several parts being added together amount to 99;

$$\therefore x+3+x+x+13+x-6+x+19=99.$$

Collecting the terms together, $5x+29=99$.

By transposition, $5x=99-29$,

$$5x=70;$$
$$\therefore x=14.$$

Hence, the several parts are 17, 14, 27, 8, 33.

(p. 24.)—Prob. 15. Two merchants entered into a speculation by which A gained £54 more than B. The whole gain was £49 less than three times the gain of B; what were their gains?

Let $x =$ B's gain in £
then $x+54 =$ A's ....

Now the sum of the gain of A and B was *the whole gain;*
$\therefore$ *the whole gain* $= 2x+54$;

But by the problem, *the whole gain* was £49 less than three times B's gain;

$\therefore$ *the whole gain* is also $= 3x-49$.

These two different expressions for *the whole gain* must necessarily be equal;

$$\therefore 3x-49 = 2x-54,$$
$$3x-2x = 54+49;$$
$$\therefore x = £103, \text{ B's gain,}$$
$$\text{and } x+54 = £157, \text{ A's}$$

Prob. 16. In dividing a lot of apples among a certain number of boys, I found that by giving 6 to each, I should have too few by 8, and by giving 4 to each boy I should have 12 remaining. How many boys were there?

Let $x =$ the number of boys;

Then, if I had given 6 apples to each boy, the number of apples given away would have been $6x$, which is a number greater than the number of apples I had by 8;

$\therefore$ *the number of apples* I had was $= 6x-8$.

Again, if I had given 4 apples to each boy, I should have given away $4x$ apples, and should then have had 12 remaining;

$\therefore$ *the number of apples* I had was $= 4x+12$.

These two expressions for *the number of apples* must necessarily be equal to each other;

$$\therefore 6x-8 = 4x+12,$$
$$6x-4x = 12+8,$$
$$2x = 20;$$
$$\therefore x = 10.$$

MULTIPLICATION. CASE I.—(p. 26.)

Ex. 5.	Ex. 6.	Ex. 7.	Ex. 8.
$4abc$	$9x^2y^2$	$-4cdx$	$-7ax^2y$
$3ac$	$-2y$	$2c$	$-2ac^2x$
$12a^2bc^2$	$-18x^2y^3$	$-8c^2dx$	$+14a^2c^2x^3y$

CASE II.

Ex. 4.

Multiply $12a^3-2a^2+4a-1$
by $3x$
Product $36a^3x-6a^2x+12ax-3x$

Ex. 5.

$9a^2x+3a-x+1$
$-x^2$
$-9a^2x^3-3ax^2+x^3-x^2$

Ex. 6. Multiply $4x^2y+3x-2y$
by $-3xy$
Product $-12x^3y^2-9x^2y+6xy^2$

(p. 27.)—Ex. 8. Multiply $a^3+3a^2b+3ab^2+b^3$
by $a+b$

$$\begin{array}{l} a^4+3a^3b+3a^2b^2+ab^3 \\ \quad +a^3b+3a^2b^2+3ab^3+b^4 \\ \hline a^4+4a^3b+6a^2b^2+4ab^3+b^4 \end{array}$$

(p. 28.)—Ex. 9. Multiply $4x^2y+3xy-1$
by $2x^2-x$

$$\begin{array}{l} 8x^4y+6x^3y-2x^2 \\ \quad -4x^3y-3x^2y+x \\ \hline 8x^4y+2x^3y-2x^2-3x^2y+x \end{array}$$

(p. 28.)—Ex. 10.

$$\begin{array}{l} x^3-x^2+x-5 \\ 2x^2+x+1 \\ \hline 2x^5-2x^4+2x^3-10x^2 \\ \quad +x^4-x^3+x^2-5x \\ \quad\quad +x^3-x^2+x-5 \\ \hline 2x^5-x^4+2x^3-10x^2-4x-5 \end{array}$$

Ex. 11.

$$\begin{array}{l} 3a^2+2ab-b^2 \\ 3a^2-2ab+b^2 \\ \hline 9a^4+6a^3b-3a^2b^2 \\ \quad -6a^3b-4a^2b^2+2ab^3 \\ \quad\quad +3a^2b^2+2ab^3-b^4 \\ \hline 9a^4 \quad * \quad -4a^2b^2+4ab^3-b^4 \end{array}$$

Ex. 12.

$$\begin{array}{l} x^3+x^2y+xy^2+y^3 \\ x-y \\ \hline x^4+x^3y+x^2y^2+xy^3 \\ \quad -x^3y-x^2y^2-xy^3-y^4 \\ \hline x^4 \quad * \quad * \quad * -y^4 \end{array}$$

Ex. 13.

$$\begin{array}{l} x^2-\frac{3}{4}x+1 \\ x^2-\frac{1}{2}x \\ \hline x^4-\frac{3}{4}x^3+x^2 \\ \quad -\frac{1}{2}x^3+\frac{3}{8}x^2-\frac{1}{2}x \\ \hline x^4-\frac{5}{4}x^3+\frac{11}{8}x^2-\frac{1}{2}x \end{array}$$

ON THE SOLUTION OF SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

Ex. 3. $6(x+3)+4x=58.$

This equation becomes, by multiplying the $x+3$ within the bracket by the 6 which is without it,

$$6x+18+4x=58,$$

or $$18+10x=58.$$

By transposition, $10x = 58 - 18$,

$$10x = 40;$$

$$\therefore x = 4$$

(p. 28.)—Ex. 4. $30(x-3)+6 = 6(x+2)$

This equation, after the multiplications have been performed, becomes

$$30x - 90 + 6 = 6x + 12,$$

$$30x - 84 = 6x + 12.$$

Transposing, $30x - 6x = 12 + 84$,

$$24x = 96;$$

$$\therefore x = 4.$$

Ex. 5. $5(x+4) - 3(x-5) = 49$.

Performing the multiplications,

$$5x + 20 - 3x + 15 = 49;$$

Collecting the terms, $2x + 35 = 49$,

$$2x = 49 - 35.$$

$$2x = 14.$$

$$\therefore x = 7.$$

Ex. 6. $4(3+2x) - 2(6-2x) = 60$.

$$12 + 8x - 12 + 4x = 60,$$

$$12x = 60,$$

$$\therefore x = 5.$$

Ex. 7. $3(x-2) + 4 = 4(3-x)$.

$$3x - 6 + 4 = 12 - 4x,$$

$$3x + 4x = 12 + 6 - 4,$$

$$7x = 14;$$

$$\therefore x = 2.$$

Ex. 8. $6(4-x) - 4(6-2x) - 12 = 0$.

$$24 - 6x - 24 + 8x - 12 = 0,$$

$$-6x + 8x - 12 = 0,$$

$$8x - 6x = 12,$$

$$2x = 12;$$

$$\therefore x = 6.$$

PROBLEMS.

(p. 30.)—PROB. 4. What number is that to which if 8 be added, twice the sum will be 24 ?

Let x denote the number;

then $x+6=$ the number, and 6 added to it,

and $2(x+6)=$ twice this sum.

But twice this sum is equal to 24;

$$\therefore 2(x+6)=24,$$
$$2x+12=24,$$
$$2x=24-12,$$
$$2x=12;$$
$$\therefore x=6.$$

PROB. 5. What two numbers are those, whose difference is 6, and if 12 be added to 4 times their sum, the whole will be 60?

Let $x=$ the less number,

then $x+6=$ the greater number;

$\therefore 2x+6=$ the sum of the numbers,

and $4(2x+6)=4$ *times* the sum of the numbers.

But 4 times the sum of the Nos. $+12=60$;

$$\therefore 4(2x+6)+12=60.$$
$$8x+24+12=60,$$
$$8x+36=60,$$
$$8x=60-36,$$
$$8x=24;$$
$$\therefore x=3, \text{ the less No.}$$
$$\text{and } x+6=9, \text{ the greater No.}$$

PROB. 6. Tea at 6*s*. per ℔. is mixed with tea at 4*s*. per ℔., and 16 ℔s. of the mixture is sold for £3 18*s*. How many ℔s. were there of each sort ?

Let $x=$ the number of lbs. at 6*s*. per lb.,

then $16-x=$ .... 4*s*. ......

$\therefore 6x=$ the price in shillings of the former kind,

and $4(16-x)=$ .... latter ......

But the sum of these prices amounts to £3 18*s.* or to 78*s.*;

$$\therefore 6x+4(16-x)=78,$$
$$6x+64-4x=78,$$
$$2x=78-64,$$
$$2x=14;$$
$$\therefore x=7 \text{ lbs.}$$
$$\text{and } 16-x=9 \text{ lbs.}$$

(p. 31.)—PROB. 7. The speed of a railway train is 24 miles an hour, and 3 hours after its departure an express train is started to run 32 miles an hour. In how many hours does the express overtake the train first started?

Let the express overtake the 1st train in x hours;
then the 1st train runs $x+3$ hours;
$\therefore 32x =$ the No. of miles run by the express train,
and $24(x+3) =$ .... 1st ......

But these two expressions, since the trains run the same distance, must be equal to each other;

$$\therefore 32x=24(x+3),$$
$$32x=24x+72,$$
$$32x-24x=72,$$
$$8x=72;$$
$$\therefore x=9 \text{ hours.}$$

(p. 32.)—PROB. 13. There are two numbers whose difference is 14, and if 9 times the less be subtracted from 6 times the greater, the remainder will be 33. What are the numbers?

Let $x =$ the less number,
then $x+14 =$ the greater number;
$\therefore 9x = 9$ times the less number,
and $6(x+14) = 6$ times the greater number.

But when 9 times the less is subtracted from 6 times the greater, the remainder is 33.

$$\therefore 6(x+14)-9x=33,$$
$$6x+84-9x=33.$$

By transposition, $9x-6x=84-33,$
$$3x=51;$$
$\therefore x=17$, the less number,
and $x+14=31$, the greater number.

PROB. 14. Two persons, A and B, lay out equal sums of money in trade; A *gains* £120, and B *loses* £80; and now A's money is *treble* of B's. What sum had each at first?

Let $x =$ the number of £ laid out by *each*,
then $x+120 =$ A's capital and gain in £,
and $x-80 =$ the residue of B's capital after losing £80, and A's money is now treble of B's;

$$\therefore x+120 = 3(x-80),$$
$$x+120 = 3x-240.$$
Transposing, $3x-x = 120+240$,
$$2x = 360;$$
$\therefore x =$ £180, the sum laid out by each.

(p. 32.)—PROB. 15. A rectangle is 8 feet long, and if it were 2 feet broader, its area would be 48 feet. Find the breadth.

Let $x =$ the breadth in feet;
then $x+2 =$ the breadth, if it were 2 feet broader;
Now the area = length multiplied by the breadth,
$= 8(x+2)$.
But by the problem, the area = 48 feet;
$$\therefore 8(x+2) = 48.$$
$$8x+16 = 48,$$
$$8x = 48, -16.$$
$$8x = 32;$$
$\therefore x = 4$ ft., the breadth required.

PROB. 16. William has 4 times as many marbles as Thomas, but, if 12 be given to each, William will then have only twice as many as Thomas. How many has each?

Let $x =$ the number Thomas has,
then $4x =$ .... William ...
$\therefore x+12 =$ the No. Thomas has, after 12 are given to him,
& $4x+12 =$ .... William ....

But William has now only *twice* as many as Thomas;
$$\therefore 4x+12 = 2(x+12),$$
$$4x+12 = 2x+24,$$
$$4x-2x = 24-12,$$
$$2x = 12;$$
$$\therefore x = 6,$$
and $\therefore$ William has 24, and Thomas has 6 marbles.

(p. 33.)—Prob. 18. Two rectangular boards are equal in area; the breadth of the one is 18 inches, and that of the other 16 inches, and the difference of their lengths 4 inches. Find the length of each, and the common area.

Let $x =$ the length in *inches* of one board,
then $x+4 =$ .... the other;
$\therefore$ the area of the 1[st] board $= 18x$,
and .... 2[d] ...... $= 16(x+4)$.

But by the problem the two areas are equal;

$$\therefore 18x = 16(x+4),$$
$$18x = 16x+64;$$
$$\therefore 18x-16x = 64,$$
$$2x = 64;$$

$\therefore x = 32$, the length of one board,
and $x+4 = 36$, ... the other ......

The area of the 1[st] board $=$ 32 in. $\times$ 18 in. $= 576$ sq. inches,
.... 2[d] ...... $= 36\times 16 = 576$ ....

Hence, it is evident, that the areas of the two boards are the same, and that their common area is 576 *square inches.*

(p. 34.)—Prob. 21. A weight of 6 ℔s. balances a weight of 24 ℔s. on a straight lever (supposed to be without weight), whose length is 20 inches; if 3 ℔s. be added to each weight, how many inches must be added to the shorter arm, in order that the lever may in its original position retain its equilibrium?

In the solution to problem 20 (vide *Algebra*), it has been found, that the arms of the lever, in its original position, are 4 and 16 inches respectively.

Let then $x =$ the No. of inches by which the shorter arm is lengthened,

And the lengths of the arms will be $x+4$ and 16 inches respectively,

And the weights corresponding to them will be 27 and 9 lbs. respectively;

Hence, in order that the lever may still retain its equilibrium, we must have the equality;

$$27(x+4) = 9\times 16,$$
$$3(x+4) = 16,$$
$$3x+12 = 16,$$
$$3x = 4;$$
$$\therefore x = 1\tfrac{1}{3} \text{ inch.}$$

(p. 34.)—Prob. 22. A garrison of 1000 men was victualled for 30 days; after 10 days it was reinforced; and then the provisions were exhausted in 5 days. Find the number of men in the reinforcement?

At the time the reinforcement arrived, the provisions remaining unconsumed were sufficient to have lasted the garrison for 20 days:

Then, if x be the number of the reinforcement, it is evident that we must have the equality;

$$(x+1000)\,5 = 1000 \times 20,$$
$$5x+5000 = 20000,$$
$$5x = 15000;$$
$$\therefore\ x = 3000, \text{ No. of the reinforcement.}$$

Prob. 23. Two triangles have each a base of 20 feet; but the altitude of one of them is 6 feet less than that of the other, and the area of the greater triangle is twice that of the less. Find their altitudes.

Let $x =$ the altitude in feet of the less triangle,
then $x+6 =$ .... greater ......
$\therefore$ the area of the less triangle $= \frac{1}{2}(20 \times x) = 10x$,
and the area of the greater ...... $= \frac{1}{2}(x+6) \times 20 = 10(x+6)$

But by the problem,

$$10\,(x+6) = 2 \times 10x;$$
$$\therefore\ x+6 = 2x;$$
$$\therefore\ x = 6 \text{ in., the altitude of the less triangle,}$$
and $x+6 = 12$ .... greater ......

Prob. 24. A and B began to play with equal sums; A won 12*s.*; then 6 times A's money was equal to 9 times B's. What had each at first?

Let $x =$ the No. of shillings each began to play with;
then $x+12 =$ A's money after *winning* 12*s.*;
and $x-12 =$ B's .... *losing* 12*s.*
$\therefore$ by the problem, $9\,(x-12) = 6\,(x+12)$,

$$9x-108 = 6x+72,$$
$$9x-6x = 72+108,$$
$$3x = 180,$$
$$\therefore\ x = 60s. = £3.$$

(p. 35.)—Prob. 25. A company settling their reckoning at a tavern, pay 4 shillings each; but observe, that if there had been five more, they would only have paid 3 shillings each. How many were there?

Let $x =$ the number of the company;
then $x+5 =$ .... if there had been 5 more;
$\therefore 4x =$ the amount the company paid in shillings,
and $3(x+5) =$ the amount paid, if there had been 5 more.

But what they *actually* paid, and what they would have paid if there had been 5 more, were the same in amount, viz., the reckoning;

$$\therefore 4x = 3(x+5),$$
$$4x = 3x+15;$$
$$\therefore x = 15, \text{ the No. of the company.}$$

(p. 35.)—PROB. 26. Two persons, A and B, at the same time set out from two towns 40 miles apart, and meet each other in 5 hours; but B walks 2 miles an hour more than A. How many miles does A walk in an hour?

Let $x =$ the number of miles A walks in an hour,
then $x+2 =$ .... B ....
$\therefore$ since they each walk 5 hours before they meet,
$5x =$ the number of miles A walks,
and $5(x+2) =$ .... B ......

But the number of miles which both A and B walk, is the distance between the two towns, or 40 miles;

$$\therefore 5x+5(x+2) = 40,$$
$$5x+5x+10 = 40,$$
$$10x+10 = 40,$$
$$10x = 40-10,$$
$$10x = 30;$$
$$\therefore x = 3 \text{ miles.}$$

DIVISION. CASE I.—(p. 36.)

Ex. 4.

Divide $25a^3c^2$ by $-5a^2c$.

$$\frac{+25a^3c^2}{-5a^2c} = -5ac.$$

Ex. 5.

Divide $-14a^3b^2c$ by $7ac$.

$$\frac{-14a^3b^2c}{+7ac} = -2a^2b^2.$$

(p. 36.)—Ex. 6. Divide $-20x^2y^2z^3$ by $-4yz$.

$$\frac{-20x^2y^2z^3}{-4yz}=+5x^2yz^2.$$

CASE II.—(p. 37.)

Ex. 3. Divide $4x^3-2x^2+2x$ by $2x$.

$$\frac{4x^3-2x^2+2x}{2x}=\frac{4x^3}{2x}-\frac{2x^2}{2x}+\frac{2x}{2x}=2x^2-x+1.$$

Ex. 4. Divide $-24a^2x^2y-3axy+6x^2y^2$ by $-3xy$.

$$\frac{-24a^2x^2y-3axy+6x^2y^2}{-3xy}=\frac{24a^2x^2y}{3xy}+\frac{3axy}{3xy}-\frac{6x^2y^2}{3xy}$$
$$=8a^2x+a-2xy.$$

Ex. 5. Divide $14ab^3+7a^2b^2-21a^2b^3+35ab^3$ by $7ab$.

$$\frac{14ab^3+7a^2b^2-21a^2b^3+35ab^3}{7ab}=\frac{14ab^3}{7ab}+\frac{7a^2b^2}{7ab}-\frac{21a^2b^3}{7ab}+\frac{35ab^3}{7ab}$$
$$=2b^2+ab-3ab^2+5b^2.$$

CASE III.—(p. 40.)

Ex. 7.

$$
\begin{array}{l}
a+b\,)\,a^4+4a^3b+6a^2b^2+4ab^3+b^4\,(\,a^3+3a^2b+3ab^2+b^3 \\
\quad a^4+\ a^3b \\
\quad \overline{\ \ *\quad 3a^3b+6a^2b^2} \\
\qquad\quad 3a^3b+3a^2b^2 \\
\qquad\quad \overline{\ \ *\quad 3a^2b^2+4ab^3} \\
\qquad\qquad\quad 3a^2b^2+3ab^3 \\
\qquad\qquad\quad \overline{\ \ *\quad ab^3+b^4} \\
\qquad\qquad\qquad\quad ab^3+b^4 \\
\qquad\qquad\qquad\quad \overline{\ \ *\quad\ *\ }
\end{array}
$$

(p. 40.)—Ex. 8.

$$a^3-3a^2x+3ax^2-x^3 \,\big)\, a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5 \,\big(\, a^2-2ax+x^2$$

$$\begin{array}{l}
a^5-3a^4x+3a^3x^2-a^2x^3 \\
\hline
* -2a^4x+7a^3x^2-9a^2x^3+5ax^4 \\
\quad -2a^4x+6a^3x^2-6a^2x^3+2ax^4 \\
\hline
\quad * + a^3x^2-3a^2x^3+3ax^4-x^5 \\
\quad\quad + a^3x^2-3a^2x^3+3ax^4-x^5 \\
\hline
\quad\quad * \quad * \quad * \quad * \\
\hline\hline
\end{array}$$

Ex. 9

$$5x^3-4x^2 \,\big)\, 25x^6-x^4-2x^3-8x^2 \,\big(\, 5x^3+4x^2+3x+2$$

$$\begin{array}{l}
25x^6-20x^5 \\
\hline
* +20x^5-x^4 \\
\quad +20x^5-16x^4 \\
\hline
\quad * +15x^4-2x^3 \\
\quad\quad +15x^4-12x^3 \\
\hline
\quad\quad * +10x^3-8x^2 \\
\quad\quad\quad +10x^3-8x^2 \\
\hline
\quad\quad\quad * \quad * \\
\hline\hline
\end{array}$$

Ex. 10.

$$a+2x \,\big)\, a^4+8a^3x+24a^2x^2+32ax^3+16x^4 \,\big(\, a^3+6a^2x+12ax^2+8x^3$$

$$\begin{array}{l}
a^4+2a^3x \\
\hline
* \; 6a^3x+24a^2x^2 \\
\quad 6a^3x+12a^2x^2 \\
\hline
\quad * \; 12a^2x^2+32ax^3 \\
\quad\quad 12a^2x^2+24ax^3 \\
\hline
\quad\quad * \; 8ax^3+16x^4 \\
\quad\quad\quad 8ax^3+16x^4 \\
\hline
\quad\quad\quad * \quad * \\
\hline\hline
\end{array}$$

(p. 41.)—Ex. 11.

$$a-x)\,a^5-x^5\,(a^4+a^3x+a^2x^2+ax^3+x^4$$
$$a^5-a^4x$$
$$*\ +a^4x-x^5$$
$$+a^4x-a^3x^2$$
$$*\ +a^3x^2-x^5$$
$$+a^3x^2-a^2x^3$$
$$*\ +a^2x^3-x^5$$
$$+a^2x^3-ax^4$$
$$*\ +ax^4-x^5$$
$$+ax^4-x^5$$
$$*\ \ *$$

Ex. 12.

$$3x^2-3x)\,6x^4+9x^2-20x\,(2x^2+2x+5-\frac{5x}{3x^2-3x}$$
$$6x^4-6x^3$$
$$*\ +6x^3+9x^2$$
$$+6x^3-6x^2$$
$$*\ +15x^2-20x$$
$$+15x^2-15x$$
$$*\ -5x$$

Ex. 13.

$$x^2-4x-5)\,9x^6-46x^5+95x^2+150x\,(9x^4-10x^3+5x^2-30x$$
$$9x^6-36x^5-45x^4$$
$$*\ -10x^5+45x^4+95x^2$$
$$-10x^5+40x^4+50x^3$$
$$*\ +5x^4-50x^3+95x^2$$
$$+5x^4-20x^3-25x^2$$
$$*\ -30x^3+120x^2+150x$$
$$-30x^3+120x^2+150x$$
$$*\ \ *\ \ *$$

(p. 41.)—Ex. 14.

$$\tfrac{4}{3}x-2\,)\,x^4-\tfrac{13}{6}x^3+x^2+\tfrac{4}{3}x-2\,(\,\tfrac{3}{4}x^3-\tfrac{1}{2}x^2+1$$

$$x^4-\tfrac{6}{4}x^3$$

$$* \quad -\tfrac{2}{3}x^3+x^2$$

$$-\tfrac{2}{3}x^3+x^2$$

$$* \quad * \quad +\tfrac{4}{3}x-2$$

$$+\tfrac{4}{3}x-2$$

$$* \quad *$$

PROBLEMS PRODUCING SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

(p. 42.)—PROB. 3. A cistern is filled in 20 minutes by 3 pipes, one of which conveys 10 gallons more, and another 5 gallons less, than the third per minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

Let x gallons flow through 3[d] pipe per min.
then $x+10$ .... 1[st] ....
and $x-5$ .... 2[d] ....
∴ taking the sum of these, we get
$3x+5=$ the galls. conveyed by all three per min.
∴ $10(3x+5)=$ .... in 20 ......

But the cistern, which is filled in 20 minutes, holds 820 gallons,

$$\therefore 20(3x+5)=820,$$
$$3x+5=41;$$
$$3x=36;$$

∴ $x=12$, galls. conveyed by 3[d] ℔ min.
$x+10=22$, ... 1[st] ......
and $x-5=7$, ... 2[d] ......

PROB. 4. A and B have the same income: A lays by $\frac{1}{5}$ of his; but B spending £60 a-year more than A, at the end of 4 years finds himself £160 in debt. What did each annually receive?

Let $5x =$ the income of each in £
then $x =$ what A lays by annually;
$\therefore 4x =$ .... spends ....
and $4x+60 =$ what B spends ....

But B is running into debt; it is clear therefore that his expenditure — his income = his debt;

$\therefore (4x+60)-5x =$ B's annual debt,
and in 4 years he incurs a debt of £160;

$$\therefore 4\left\{(4x+60)-5x)\right\} = 160,$$
$$4x+60-5x = 40,$$
$$60-x = 40;$$
$$\therefore x = 20,$$
and $5x =$ £100, the income of each.

Prob. 5. A met two beggars, B and C, and having a certain sum in his pocket, gave B $\frac{1}{6}$th of it, and C $\frac{3}{5}$ths of the remainder: A now had $20d.$ left; what had he at first?

Assume $30x =$ the No. of $d.$, which A had at first,
then $\frac{1}{6}$ of $30x$, or $5x =$ .... A gave to B.

Now, the money which A gave deducted from what he had is the remainder;

$$\therefore \text{the remainder} = 30x-5x = 25x.$$

Of this remainder A gave $\frac{3}{5}$ths to C, or $\frac{3}{5}$ths of $25x = 15x$;
then A had left $25x-15x = 10x$.

But what he had left is by the problem $= 20d.$

$$\therefore 10x = 20,$$
and $30x = 60d. = 5s.$

Prob. 6. A person has two horses, and a saddle worth £60; if the saddle be put on the first horse, his value will become double that of the second; but if it be put on the second, the value will become triple that of the first. What is the value of each horse?

Let $2x =$ the value of the 1st horse in £
then $2x+60 =$ double the value of the 2d horse in £
$\therefore x+30 =$ the value of the 2d horse in £.

But the saddle being put on the 2d horse makes his value triple that of the 1st;

$\therefore (x+30)+60 =$ triple the value of the 1st horse:
But $2x \times 3 =$ ....

Equating therefore these two equal values,

$$2x \times 3 = (x+30)+60,$$
$$6x = x+90,$$
$$5x = 90,$$
$$x = 18;$$
$$\therefore 2x = £36, \text{ the value of the 1}^{\text{st}}\text{ horse,}$$
$$\text{and } x+30 = £48, \ldots\ldots\ldots\ldots\ldots\ldots \text{2}^{\text{d}} \ldots\ldots$$

(p. 43.)—Prob. 8. A person at play lost a third of his money, and then won 4*s.*; he again lost a fourth of his money, and then won 13*s.*; lastly, he lost an eighth of what he then had, and found he had 28*s.* left. What had he at first?

In the solution of this problem, the *coefficient* of x, which is to be assumed, must be a *multiple* of 3, 4, and 8; since the fractions $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{8}$, are involved in the problem. The *least* multiple of 3, 4, and 8, is 24. Assume, therefore,

$24x$ = the number of shillings he had at first.

First of all he lost $\frac{1}{3}$ of his money; consequently, he had $\frac{2}{3}$ of it left;

$\therefore$ what he had left was $= \frac{2}{3}$ of $24x = 16x$.

After this loss, he won 4*s.*

and $\therefore$ he then had a sum in shillings $= 16x+4$.

He again lost $\frac{1}{4}$ of his money, and $\therefore$ had $\frac{3}{4}$ of it left;

$\therefore$ he now had left a sum $= \frac{3}{4}(16x+4) = 12x+3$.

He now won 13*s.*, and

$\therefore$ he again had a sum $= 12x+16$.

Finally, he lost $\frac{1}{8}$ of this last sum of money, and $\therefore$ he had $\frac{7}{8}$ of it left;

$\therefore$ he now had remaining a sum in shillings $= \frac{7}{8}(12x+16)$.

But the money which in the end he had left, is stated in the problem to be 28*s.* Hence, we have the equation,

$$\frac{7}{8}(12x+16) = 28,$$
$$12x+16 = 4 \times 8,$$
$$12x = 16;$$
$$\therefore 24x = 32s. = £1\ 12s.$$

CHAPTER III.

ON ALGEBRAIC FRACTIONS.

To reduce a mixed Quantity to an improper Fraction.

(p. 45.)—Ex. 4. Reduce $4ab+\frac{2c}{3a}$ to an improper fraction.

Here $4ab+\frac{2c}{3a}=\frac{4ab\times 3a+2c}{3a}=\frac{12a^2b+2c}{3a}$.

Ex. 5. Reduce $3b^2-\frac{4a}{5x}$ to an improper fraction.

Here $3b^2-\frac{4a}{5x}=\frac{3b^2\times 5x-4a}{5x}=\frac{15b^2x-4a}{5x}$.

Ex. 6. Reduce $a-x+\frac{a^2-ax}{x}$ to an improper fraction.

Here $a-x+\frac{a^2-ax}{x}=\frac{ax-x^2+a^2-ax}{x}$

$=\frac{a^2-x^2}{x}$, (since $ax-ax=0$.)

Ex. 7. Reduce $3x^2-\frac{4x-9}{10}$ to an improper fraction.

Here $3x^2-\frac{4x-9}{10}=\frac{3x^2\times 10-(4x-9)}{10}=\frac{30x^2-4x+9}{10}$.

To reduce an improper Fraction to a mixed Quantity.

(p. 46.)— Ex. 3. Reduce $\frac{4x^2-5a}{2x}$ to a mixed quantity.

Here $\frac{4x^2}{2x}=2x$ is the *integral* part,

and $\frac{-5a}{2x}=-\frac{5a}{2x}$ is the *fractional* part,

∴ the mixed quantity required $=2x-\frac{5a}{2x}$.

(p. 46.)—Ex. 4. Reduce $\frac{12a^2+4a-3c}{4a}$ to a mixed quantity.

$$\frac{12a^2+4a-3c}{4a}=\frac{12a^2}{4a}+\frac{4a}{4a}-\frac{3c}{4a}=3a+1-\frac{3c}{4a}.$$

Ex. 5. Reduce $\frac{10x^2y+3x^3-2b^2}{x^2}$ to a mixed quantity.

$$\frac{10x^2y+3x^3-2b^2}{x^2}=\frac{10x^2y}{x^2}+\frac{3x^3}{x^2}-\frac{2b^2}{x^2}=10y+3x-\frac{2b^2}{x^3},$$

which is the mixed quantity required.

To reduce Fractions to a common Denominator.

(p. 47.)—Ex. 4.

Reduce $\frac{3x}{5}$, $\frac{4bx}{3a}$, and $\frac{5x^2}{a}$, to a common denominator.

$$\left.\begin{array}{l} 3x \times 3a \times a = 9a^2x \\ 4bx \times 5 \times a = 20abx \\ 5x^2 \times 5 \times 3a = 75ax^2 \end{array}\right\} \text{the new numerators,}$$

$$5 \times 3a \times a = 15a^2, \text{ the new denominator;}$$

hence the fractions required are $\frac{9a^2x}{15a^2}$, $\frac{20abx}{15a^2}$, $\frac{75ax^2}{15a^2}$.

Ex. 5.

Reduce $\frac{2x+3}{x}$ and $\frac{5x+1}{3}$ to a common denominator.

$$\text{Here } \left.\begin{array}{l} (2x+3)\times 3 = 6x+9 \\ (5x+1)\times x = 5x^2+x \end{array}\right\} \text{the new numerators,}$$

$$x \times 3 = 3x, \quad \text{the new denominator;}$$

hence the required fractions are $\frac{6x+9}{3x}$, and $\frac{5x^2+x}{3x}$.

Ex. 6.

Reduce $\frac{4x^2+2x}{5}$, $\frac{3x^2}{4a}$, $\frac{2x}{3b}$, to a common denominator

$$\begin{aligned}(4x^2+2x)\times 4a\times 3b &= 48abx^2+24abx\\ 3x^2\times 5\ \times 3b &= 45bx^2\\ 2x\ \times 5\ \times 4a &= 40ax\\ \hline 5\ \times 4a\times 3b &= 60ab\end{aligned}$$

$\therefore \dfrac{48abx^2+24abx}{60ab}, \dfrac{45bx^2}{60ab}, \dfrac{40ax}{60ab}$, are the required fractions.

(p. 47.)—Ex. 7.

Reduce $\dfrac{7x^2-1}{2x}$ and $\dfrac{4x^2-x+2}{2a^2}$ to a common denominator.

$$\begin{aligned}(7x^2-1)\times 2a^2 &= 14a^2x^2-2a^2\\ (4x^2-x+2)\times 2x &= 8x^3-2x^2+4x\\ \hline 2x\times 2a^2 &= 4a^2x\end{aligned}$$

$\therefore$ the required fractions are $\dfrac{14a^2x^2-2a^2}{4a^2x}$, and $\dfrac{8x^3-2x^2+4x}{4a^2x}$.

To reduce a Fraction to its lowest Terms.

(p. 48.)—Ex. 4. Reduce $\dfrac{10x^3}{15x^2}$ to its lowest terms.

The *greatest common divisor*, that the numerator and the denominator admit of, is $5x^2$; $\therefore$ dividing them both by $5x^2$, the fraction becomes $\dfrac{2x}{3}$, which is in the lowest terms.

Ex. 5. Reduce $\dfrac{3abx^2}{6ax}$ to its lowest terms.

The greatest common divisor is $3ax$; $\therefore$ after division, $\dfrac{3abx^2}{6ax}=\dfrac{bx}{2}$, which is in the lowest terms.

Ex. 6. Reduce $\dfrac{14x^2y^2-21x^3y^2}{7x^3y}$ to its lowest terms.

Here the greatest common divisor is $7x^2y$;

$$\therefore \frac{14x^2y^2-21x^3y^2}{7x^2y}=2y-3xy,$$

$$\text{and } \frac{7x^3y}{7x^2y}=x.$$

Hence, the fraction in its lowest terms is $\frac{2y-3xy}{x}$.

(p. 48.)—Ex. 7.

Reduce $\frac{51x^3-17x^2+34x}{17x^3}$ to its lowest terms.

The greatest quantity, which will divide both the numerator and the denominator without remainders, is $17x$.

$$\therefore \frac{51x^3-17x^2+34x}{17x}=3x^2-x+2,$$

$$\text{and } \frac{17x^3}{17x}=x^2$$

Hence the fraction in its lowest terms is $\frac{3x^2-x+2}{x^2}$.

ON THE ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION, OF FRACTIONS.

To add Fractions together.

(p. 50.)—Ex. 4. Add $\frac{3x}{7}$, $\frac{5x}{9}$, and $\frac{4x}{11}$, together.

$$3x\times9\times11=297x$$
$$5x\times7\times11=385x$$
$$4x\times7\times\ 9=252x$$
$$7\ \times9\times11=693$$

Hence the sum of fractions $=\frac{297x}{693}+\frac{385x}{693}+\frac{252x}{693}=\frac{934x}{693}$.

Ex. 5. Add $\frac{3a^2}{2b}$, $\frac{2a}{5}$, and $\frac{3b}{7a}$, together.

Here
$$\begin{aligned} 3a^2\times 5\times 7a &= 105a^3 \\ 2a\times 2b\times 7a &= 28a^2b \\ 3b\times 2b\times 5 &= 30b^2 \\ \hline 2b\times 5\times 7a &= 70ab \end{aligned}$$

$\therefore$ the sum $=\frac{105a^3}{70ab}+\frac{28a^2b}{70ab}+\frac{30b^2}{70ab}=\frac{105a^3+28a^2b+30b^2}{70ab}$.

(p.50.)—Ex. 6. Add $\frac{2x+1}{3}$, $\frac{4x+2}{5}$, and $\frac{x}{7}$, together.

$$\begin{aligned} (2x+1)\times 5\times 7 &= 70x+35 \\ (4x+2)\times 3\times 7 &= 84x+42 \\ x\times 3\times 5 &= 15x \\ \hline 3\times 5\times 7 &= 105 \end{aligned}$$

$\therefore$ the req^d. sum $=\frac{70x+35}{105}+\frac{84x+42}{105}+\frac{15x}{105}=\frac{169x+77}{105}$.

Ex. 7. Add $\frac{5a^2+b}{3b}$ and $\frac{4a^2+2b}{5b}$ together.

$$\begin{aligned} (5a^2+b)\times 5b &= 25a^2b+5b^2 \\ (4a^2+2b)\times 3b &= 12a^2b+6b^2 \\ \hline 3b\times 5b &= 15b^2 \end{aligned}$$

$\therefore$ the sum $=\frac{25a^2b+5b^2}{15b^2}+\frac{12a^2b+6b^2}{15b^2}=\frac{37a^2b+11b^2}{15b^2}$

$=$ (after dividing num^r. and den^r. by b) $\frac{37a^2+11b}{15b}$.

Ex. 8. Add $\frac{2x-5}{3}$ and $\frac{x-1}{2x}$ together.

$$\begin{aligned} (2x-5)\times 2x &= 4x^2-10x \\ (x-1)\times 3 &= 3x-3 \\ \hline 3\times 2x &= 6x \end{aligned}$$

$\therefore$ the sum of fractions $=\frac{4x^2-10x}{6x}+\frac{3x-3}{6x}+\frac{4x^2-7x-3}{6x}$.

Ex. 9. Add $\frac{x}{x-3}$ and $\frac{x}{x+3}$ together.

$$\begin{array}{r} x\times(x+3)=x^2+3x \\ x\times(x-3)=x^2-3x \\ \hline (x-3)\times(x+3)=x^2-9 \end{array}$$

$\therefore$ the sum of the fractions $=\frac{x^2+3x}{x^2-9}+\frac{x^2-3x}{x^2-9}=\frac{2x^2}{x^2-9}$.

(p. 50.)—Ex. 10. Add $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$ together.

$$\begin{array}{r} (a+b)\times(a+b)=a^2+2ab+b^2 \\ (a-b)\times(a-b)=a^2-2ab+b^2 \\ \hline (a-b)\times(a+b)=a^2-b^2 \end{array}$$

$\therefore$ the sum $=\frac{a^2+2ab+b^2}{a^2-b^2}+\frac{a^2-2ab+b^2}{a^2-b^2}=\frac{2a^2+2b^2}{a^2-b^2}$.

To subtract Fractional Quantities.

(p. 51.)—Ex. 5. Subtract $\frac{4x}{5}$ from $\frac{9x}{10}$.

$$\begin{array}{r} 4x\times10=40x \\ 9x\times\ 5=45x \\ \hline 5\ \times10=50 \end{array}$$

$\therefore$ the difference of the fractions $=\frac{45x}{50}-\frac{40x}{50}=\frac{5x}{50}=\frac{x}{10}$.

(p. 52.)—Ex. 6. Subtract $\frac{5x+1}{7}$ from $\frac{21x+3}{4}$.

$$\begin{array}{r} (5x+1)\times4=\ 20x+\ 4 \\ (21x+3)\times7=147x+21 \\ \hline 7\times4=28 \end{array}$$

$\therefore$ the difference $=\frac{147x+21}{28}-\frac{20x+4}{28}=\frac{127x+17}{28}$.

Ex. 7. Subtract $\frac{3x+1}{x+1}$ from $\frac{4x}{5}$.

$$\begin{array}{r} (3x+1)\times5=15x+5 \\ 4x\times(x+1)=\ 4x^2+4x \\ \hline (x+1)\times5\ =\ 5x+5 \end{array}$$

$$\therefore \text{ the difference} = \frac{4x^2+4x}{5x+5} - \frac{15x+5}{5x+5} = \frac{4x^2-11x-5}{5x+5}.$$

(p. 52.)—Ex. 8. Subtract $\frac{2x-3}{3x}$ from $\frac{4x+2}{3}$.

$$(2x-3)\times 3 = 6x-9$$
$$(4x+2)\times 3x = 12x^2+6x$$
$$3x\times 3 = 9x$$

$$\therefore \text{ the difference} = \frac{12x^2+6x}{9x} - \frac{6x-9}{9x} = \frac{12x^2+9}{9x} = \frac{4x^2+3}{3x}.$$

Ex. 9. Subtract $\frac{1}{a+b}$ from $\frac{1}{a-b}$.

$$1\times(a-b) = a-b$$
$$1\times(a+b) = a+b$$
$$(a-b)\times(a+b) = a^2-b^2$$

$$\therefore \text{ the difference} = \frac{a+b}{a^2-b^2} - \frac{a-b}{a^2-b^2} = \frac{a+b-a+b}{a^2-b^2} = \frac{2b}{a^2-b^2}.$$

Ex. 10. Subtract $\frac{3x-7}{8}$ from $\frac{4x}{7}$.

$$(3x-7)\times 7 = 21x-49$$
$$4x\times 8 = 32x$$
$$8\times 7 = 56$$

$$\therefore \text{ the difference of fractions} = \frac{32x}{56} - \frac{21x-49}{56} = \frac{11x+49}{56}.$$

To multiply Fractional Quantities.

(p. 53.)—Ex. 5. Multiply $\frac{2x}{x-1}$ by $\frac{3x}{7}$.

$$\text{Here } \frac{2x}{x-1}\times\frac{3x}{7} = \frac{2x\times 3x}{(x-1)\times 7} = \frac{6x^2}{7x-7}.$$

(p. 53.)—Ex. 6. Multiply $\frac{3x^2-x}{5}$ by $\frac{10}{2x^2-4x}$.

$$\frac{3x^2-x}{5}\times\frac{10}{2x^2-4x}=\frac{(3x-1)x\times2\times5}{5\times2x\,(x-2)}$$
$$=\frac{(3x-1)\times1}{1\times(x-2)}=\frac{3x-1}{x-2}.$$

Ex. 7. Multiply $\frac{2a}{a-b}$ by $\frac{a^2-b^2}{8}$

$$\frac{2a}{a-b}\times\frac{a^2-b^2}{8}=\frac{2a\times(a+b).(a-b)}{(a-b)\times4\times2}$$
$$=\frac{a\times(a+b)}{4}=\frac{a^2+ab}{4}.$$

Ex. 8. Multiply $\frac{3x^2}{5x-10}$ by $\frac{15x-30}{2x}$.

$$\frac{3x^2}{5x-10}\times\frac{15x-30}{2x}=\frac{3x\times x\times3\,(5x-10)}{(5x-10)\times2\times x}$$
$$=\frac{3x\times3}{2}$$
$$=\frac{9x}{2}.$$

On the Division of Fractions.

(p. 54.)—Ex. 4. Divide $\frac{4x}{7}$ by $\frac{9x}{5}$.

$$\frac{4x}{7}\div\frac{9x}{5}=\frac{4x}{7}\times\frac{5}{9x}=\frac{4\times5}{7\times9}=\frac{20}{63}.$$

Ex. 5. Divide $\frac{4x+2}{3}$ by $\frac{2x+1}{5x}$.

$$\frac{4x+2}{3}\div\frac{2x+1}{5x}=\frac{2\,(2x+1)}{3}\times\frac{5x}{2x+1}=\frac{10x}{3}.$$

(p. 54.)—Ex. 6. Divide $\frac{x^2-9}{5}$ by $\frac{x+3}{4}$.

$$\frac{x^2-9}{5} \div \frac{x+3}{4} = \frac{(x+3).(x-3)}{5} \times \frac{4}{x+3} = \frac{4x-12}{5}.$$

Ex. 7. Divide $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$.

$$\frac{9x^2-3x}{5} \div \frac{x^2}{5} = \frac{x(9x-3)}{5} \times \frac{5}{x^2} = \frac{9x-3}{x}.$$

SIMPLE EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

(p. 56.)—Ex. 4. Let $\frac{2x}{3} + \frac{x}{4} = 22$.

Multiply each side by 3, and $2x + \frac{3x}{4} = 66$,

.... 4 ... $8x+3x=264$,

$$11x=264;$$

$$\therefore x=24.$$

Ex. 5. Let $\frac{7x}{4} - \frac{5x}{6} = \frac{55}{6}$.

Multiply each side by 4, and $7x - \frac{20x}{6} = \frac{220}{6}$,

.... 6, ... $42x-20x=220$,

$$22x=220;$$

$$\therefore x=10.$$

Ex. 6. Let $\frac{x}{2} + \frac{x}{3} = 31 - \frac{x}{5}$.

The least common multiple of 2, 3, 5, is 30; ∴ multiplying each side of the equation by 30, we get

$$\frac{30x}{2} + \frac{30x}{3} = 930 - \frac{30x}{5},$$

$$\text{or } 15x+10x=930-6x,$$
$$25x=930-6x,$$
$$25x+6x=930,$$
$$31x=930;$$
$$\therefore x=30.$$

(p. 58.)—Ex. 7. Let $\frac{2x}{5}-\frac{x}{6}+\frac{x}{2}=44.$

The *least com. mult.* of 5, 6, 2, is 30; ∴ mult. each side by 30.

$$12x-5x+15x=1320,$$
$$22x=1320;$$
$$\therefore x=60.$$

(p. 59.)—Ex. 8. $x+\frac{x}{2}+\frac{x}{3}=11,$

Multiply by 6, and $6x+3x+2x=66,$

$$11x=66;$$
$$\therefore x=6.$$

Ex. 9. $\frac{x}{5}+\frac{x}{4}+\frac{x}{3}=\frac{x}{2}+17.$

Multiply *both sides* of the equation by 60, which is the *least common multiple* of 5, 4, 3, 2, and

$$12x+15x+20x=30x+1020,$$
$$47x=30x+1020.$$

By transposition, $47x-30x=1020,$

$$17x=1020;$$
$$\therefore x=60.$$

Ex. 10. $4x-20=\frac{3x}{7}+\frac{110}{7}$

Multiply by 7, and $28x-140=3x+110.$

And, transposing, $28x-3x=110+140,$

$$25x=250;$$
$$\therefore x=10.$$

Ex. 11. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{1}{2}.$

Multiply by 12, and $6x+4x-3x=6,$

$$7x=6.$$
$$\therefore x=\frac{6}{7}$$

(p. 59.)—Ex. 12. $3x+\frac{1}{9}=\frac{x+3}{3}$.

Multiply by 9, and $27x+1=3x+9$.
By transposition, $27x-3x=9-1$,
$24x=8$.
Divide both sides by 8, and $3x=1$;
$\therefore x=\frac{1}{3}$.

Ex. 13. $\frac{3x}{7}-5=29-2x$.

Multiplying by 7, $3x-35=203-14x$.
By transposition, $3x+14x=203+35$,
$17x=238$;
$\therefore x=14$.

Ex. 14. $6x-\frac{3x}{4}-9=5x$.

Multiply by 4, then $24x-3x-36=20x$,
$21x-36=20x$.
$\therefore$ by transposition, $21x-20x=36$,
and $x=36$.

Ex. 15. $2x-\frac{x+3}{3}+15=\frac{12x+26}{5}$.

Multiply by 15, and $30x-5x-15+225=36x+78$,
$25x+210=36x+78$.
And transposing, $36x-25x=210-78$,
$11x=132$;
$\therefore x=12$.

Ex. 16. $\frac{x-2}{2}+\frac{x}{3}=20-\frac{x-6}{2}$.

$3x-6+2x=120-3x+18$,
$5x-6=138-3x$,
$5x+3x=138+6$,
$8x=144$;
$\therefore x=18$.

Ex. 17. $5x-\frac{2x-1}{3}+1=3x+\frac{x+2}{2}+7$.

$$30x-4x+2+6=18x+3x+6+42,$$
$$26x+8=21x+48,$$
$$26x-21x=48-8,$$
$$5x=40;$$
$$\therefore x=8.$$

Ex. 18. $2ax+b=3cx+4a.$

By transposition, $2ax-3cx=4a-b,$

or $(2a-3c)\,x=4a-b;$

$$\therefore x=\frac{4a-b}{2a-3c}.$$

(p. 61.)—Ex. 23. $\dfrac{3x-3}{4}-\dfrac{3x-4}{3}=5\tfrac{1}{3}-\dfrac{27+4x}{9}.$

Multiply by 36, and $27x-27-36x+48=192-108-16x.$

Coll[g]. the terms on each side $-9x+21=84-16x.$

By transposition, $16x-9x=84-21,$

$$7x=63;$$
$$\therefore x=9.$$

Ex. 24. $\dfrac{9x+20}{36}=\dfrac{4x-12}{5x-4}+\dfrac{x}{4}.$

Multiplying by 36, $9x+20=\dfrac{144x-432}{5x-4}+9x,$

$$20=\frac{144x-432}{5x-4}.$$

Mult. by $5x-4$, and $100x-80=144x-432,$

$$144x-100x=432-80,$$
$$44x=352;$$
$$\therefore x=8.$$

Ex. 25. $\dfrac{20x+36}{25}+\dfrac{5x+20}{9x-16}=\dfrac{4x}{5}+\dfrac{86}{25}.$

Mult. by 25, and $20x+36+\dfrac{125x+500}{9x-16}=20x+86.$

$$\frac{125x+500}{9x-16}=50,$$
$$125x+500=450x-800,$$
$$450x-125x=500+800,$$
$$325x=1300;$$
$$\therefore x=4.$$

Ex. 26. $\frac{2x+8\frac{1}{2}}{9}-\frac{13x-2}{17x-32}+\frac{x}{3}=\frac{7x}{12}-\frac{x+16}{36}$.

Mult. by 36, and $8x+34-\frac{468x-72}{17x-32}+12x=21x-x-16$.

$$50=\frac{468x-72}{17x-32},$$
$$850x-1600=468x-72,$$
$$850x-468x=1600-72,$$
$$382x=1528;$$
$$\therefore x=4.$$

Ex. 27. $4(5x-3)-64(3-x)-3(12x-4)=96$.

By performing the multiplications, which are indicated, the equation becomes

$$20x-12-192+64x-36x+12=96,$$
$$48x-192=96,$$
$$48x=288;$$
$$\therefore x=6.$$

Ex. 28. $10\left(x+\frac{1}{2}\right)-6x\left(\frac{1}{x}-\frac{1}{3}\right)=23$.

$$10x+5-6+2x=23,$$
$$12x-1=23,$$
$$12x=24;$$
$$\therefore x=2.$$

Ex. 29. $\frac{30+6x}{x+1}+\frac{60+8x}{x+3}=14+\frac{48}{x+1}$

Mult. by $x+1$, and $30+6x+\frac{60+68x+8x^2}{x+3}=14x+14+48$.

By transposition, $\frac{60+68x+8x^2}{x+3}=8x+32$.

Multiply by $x+3$, and $60+68x+8x^2=8x^2+56x+96$

$$68x-56x=96-60,$$
$$12x=36;$$
$$\therefore x=3.$$

PROBLEMS.

(p. 63.)—PROB. 5. What number is that to which, if I add 20, and from $\frac{2}{3}$ds of this sum I subtract 12, the remainder shall be 10?

Let x be the number;

then $x+20=$ the number, with 20 added to it,

and $\frac{2(x+20)}{3}=\frac{2}{3}$ds of this sum.

But 12 being subtracted from $\frac{2}{3}$ds of this sum, the remainder shall be 10:

$$\therefore \frac{2(x+20)}{3}-12=10.$$

$$\frac{2(x+20)}{3}=22,$$

$$2x+40=66,$$

$$2x=26;$$

$$\therefore x=13.$$

PROB. 6. What number is that, of which if I add $\frac{1}{3}$d, $\frac{1}{4}$th, and $\frac{2}{7}$ths, together, the sum shall be 73?

Let x be the number;

then $\frac{x}{3}=\frac{1}{3}$d the number,

$\frac{x}{4}=\frac{1}{4}$th ....

$\frac{2x}{7}=\frac{2}{7}$ths ....

Hence, by the conditions of the problem, we have

$$\frac{x}{3}+\frac{x}{4}+\frac{2x}{7}=73.$$

Multiply by 84, and $28x+21x+24x=73\times84,$

$$73x=73\times84;$$

$$\therefore x=84.$$

PROB. 7. What number is that whose $\frac{1}{3}$d part exceeds its $\frac{1}{5}$th part by 72?

Let x denote the number;

then $\frac{x}{3}=$ its $\frac{1}{3}$d part,

and $\frac{x}{5}=$ its $\frac{1}{5}$th ......

Hence $\frac{x}{3}-\frac{x}{5}=72$,

$5x-3x=1080$,

$2x=1080$;

$\therefore x=540$.

Prob. 8. There are two numbers whose sum is 37, and if 3 times the less be subtracted from 4 times the greater, and this difference divided by 6, the quotient will be 6. What are the numbers?

Let $x=$ the greater number;

then $37-x=$ the less ....

$\therefore 4x=4$ times the greater number,

and $3(37-x)=3$ .... less ....

Hence, by the conditions of the problem,

$$\frac{4x-3(37-x)}{6}=6.$$

$4x-111+3x=36$,

$7x=147$;

$\therefore x=21$, the greater number,

and $37-x=16$, the less ....

Prob. 9. There are two numbers whose sum is 49; and if $\frac{1}{7}$th of the less be subtracted from $\frac{1}{5}$th of the greater, the remainder will be 5. What are the numbers?

Let $x=$ the greater number;

then $49-x=$ the less ....

$\therefore \frac{x}{5}=\frac{1}{5}$th of the greater number,

and $\frac{49-x}{7}=\frac{1}{7}$th ...... less ....

$\therefore$ by the problem, $\frac{x}{5}-\frac{49-x}{7}=5$,

$$7x-245+5x=175,$$
$$12x=175+245,$$
$$12x=420;$$
$$\therefore x=35, \text{ the greater number,}$$
$$\text{and } 49-x=14, \text{ the less } \ldots\ldots\ldots$$

(p. 63.)—Prob. 10. To divide the number 72 into three parts, so that $\frac{1}{2}$ the *first* part shall be equal to the *second*, and $\frac{3}{5}$ths of the *second* part equal to the third.

Assume $10x=$ the 1st part,
then $5x= \ldots$ 2d ......
and $3x= \ldots$ 3d ......

$$\therefore 10x+5x+3x=72,$$
$$18x=72;$$
$$\therefore x=4.$$

Hence $10x=40$, the 1st part,
$5x=20$, ... 2d ......
and $3x=12$, ... 3d ......

Prob. 11. A person after spending $\frac{1}{5}$th of his income *plus* £10, had then remaining $\frac{1}{2}$ of it *plus* £35. Required his income.

Let $x=$ his income in £

then $\frac{x}{5}+10=$ what he spent of it;

$\therefore x-\left(\frac{x}{5}+10\right)=$ what he had left.

But this is by the problem $=\frac{1}{2}$ his income $+$ £35,

$$\therefore x-\frac{x}{5}-10=\frac{x}{2}+35,$$
$$10x-2x-100=5x+350,$$
$$3x=450;$$
$$\therefore x=\text{£}150, \text{ his income.}$$

Prob. 12. A gamester at *one sitting* lost $\frac{1}{5}$th of his money, and then won 10 shillings; at a *second* he lost $\frac{1}{3}$d of the remainder, and then won 3 shillings; after which he had 3 guineas left. What money had he at first;

Let $x=$ the money *in shillings* he had at first;

Then, since at the 1st *sitting* he lost $\frac{1}{5}$th of his money, he had $\frac{4}{5}$ths of it left, or $\frac{4x}{5}$: he now won 10*s.*, and $\therefore$ the money he now had, was $=\frac{4x}{5}+10$.

At the 2d *sitting* he lost $\frac{1}{3}$d of this last sum, and $\therefore$ he had left $\frac{2}{3}$ds of it, or $\frac{2}{3}\left(\frac{4x}{5}+10\right)$. He again won 3*s*;

$\therefore$ he had left after the 2d sitting a sum $=\frac{2}{3}\left(\frac{4x}{5}+10\right)+3$.

But, by the problem, this sum $=3$ guineas, or 63*s*;

$$\therefore \frac{2}{3}\left(\frac{4x}{5}+10\right)+3=63,$$

$$\frac{2}{3}\left(\frac{4x}{5}+10\right)=60,$$

$$\frac{4x}{5}+10=90,$$

$$\frac{4x}{5}=80;$$

$$\therefore x=100s.=£5.$$

(p. 63.)—PROB. 13. Divide the number 90 into four such parts, that the first *increased* by 2, the second *diminished* by 2, the third *multiplied* by 2, and the fourth *divided* by 2, may all be equal to the same quantity.

Assume $x-2=$ the 1st part,
$x+2=$ ... 2d ......
$\frac{x}{2}=$ ... 3d ......
and $2x=$ ... 4th

$$\therefore x-2+x+2+\frac{x}{2}+2x=90,$$

$$\frac{x}{2}+4x=90,$$

$$x+8x=180,$$

$$9x=180;$$

$$\therefore x=20.$$

Hence the parts are 18, 22, 10, and 40.

(p. 63.)—PROB. 14. A merchant has two kinds of tea, one worth 9*s.* 6*d.* per lb., the other 13*s.* 6*d.* How many lbs. of each must he take to form a chest of 104 lbs., which shall be worth £56.

Let $x =$ the No. of lbs. at 13*s.* 6*d.* per lb.

then $104-x =$ .... 9*s.* 6*d.* ......

$\therefore 27x =$ the price *in sixpences* of the 1st kind,

$19(104-x) =$ .... 2d ......

Now the value of both kinds is £56, or 2240 sixpences;

$$\therefore 27x+19(104-x) = 2240,$$
$$27x+1976-19x = 2240,$$
$$8x = 2240-1976,$$
$$8x = 264;$$

$\therefore x = 33$, No. of lbs. at 13*s.* 6*d.*

and $104-x = 71$, ... 9*s.* 6*d.*

(p. 64.)—PROB. 16. A man and his wife usually drank a cask of beer in 10 days, but when the man was absent it lasted the wife 30 days; how long would the man alone take to drink it?

Let $x =$ the No. of days the man took to drink the beer, and if the beer in the cask be represented by 1;

then $\frac{1}{10} =$ the beer drunk by the man and wife in 1 day,

and $\frac{1}{x} =$ .... man ......

$\therefore \frac{1}{10}-\frac{1}{x} =$ .... the wife ......

Now the beer drunk by the wife in 1 day, multiplied by 30, is equal to the beer in the cask;

$$\therefore \left(\frac{1}{10}-\frac{1}{x}\right)30 = 1,$$
$$3-\frac{30}{x} = 1,$$
$$\frac{30}{x} = 2;$$
$$\therefore x = 15 \text{ days.}$$

Prob. 17. A cistern has 3 pipes; two of which will fill it in 3 and 4 hours respectively, and the third will empty it in 6 hours; in what time will the cistern be full, if they be all set a-running at once?

Let $x =$ the No. of hours in which the cistern is filled; and, if 1 represent the contents of the cistern,

then $\frac{1}{3} =$ the part of it filled by the 1st pipe in 1 *hour*,

$\frac{1}{4} =$ .... 2d ....

and $\frac{1}{6} =$ .... emptied by the 3d pipe ....

Therefore, since the two first pipes have a tendency to fill, and the 3d to empty the cistern,

$\frac{1}{3}+\frac{1}{4}-\frac{1}{6}$ = the part of it *actually* filled in 1 *hour*,

and $\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{6}\right)x =$ .... *x hours*.

But the cistern, which is denoted by 1, is filled in x hours;

$$\therefore \left(\frac{1}{3}+\frac{1}{4}-\frac{1}{6}\right)x = 1,$$
$$(4+3-2)\,x = 12,$$
$$5x = 12;$$
$$\therefore x = \frac{12}{5} = 2 \text{ hours } 24 \text{ min.}$$

(p. 65.)—Prob. 19. A market-woman bought a certain number of apples at two a penny, and as many at three a penny, and sold them at the rate of five for two-pence; after which she found, that instead of making her money again as she expected, she lost fourpence by the whole business. How much money had she laid out?

Let $x =$ the No. of apples in each lot that she bought;

The price of each apple in the 1st lot $= \frac{1}{2}d.$

.... 2d ... $= \frac{1}{3}d.$

and *the price* at which she sold each apple $= \frac{2}{5}d.$

Therefore, she gave for the apples $\frac{1}{2}x + \frac{1}{3}x = \left(\frac{1}{2}+\frac{1}{3}\right)x,$

and she sold them for $\frac{2}{5} \times 2x = \frac{4x}{5}$.

Hence, since in selling them she lost 4*d*., we have the equation,

$$\left(\frac{1}{2}+\frac{1}{3}\right)x = \frac{4x}{5}+4,$$

$$\frac{5}{6}x = \frac{4x}{5}+4,$$

$$25x = 24x+120;$$

$$\therefore x = 120 \text{ apples.}$$

Now the price of 120 apples at $\frac{1}{2}$*d*., and of 120 more at $\frac{1}{3}$*d*. each, amounts to 100*d*., or 8*s*. 4*d*., the money which she laid out.

(p. 66.)—Prob. 21. A lady bought a hive of bees, and found that the price came to 2*s*. more than $\frac{3}{4}$ths and $\frac{1}{5}$th of the price. Find the price.

Let x denote the price in *shillings;*

then $\frac{3x}{4} = \frac{3}{4}$ths of the price,

and $\frac{x}{5} = \frac{1}{5}$th ....

Hence, by the problem, $x - \frac{3x}{4} - \frac{x}{5} = 2,$

$$20x - 15x - 4x = 40;$$

$$\therefore x = 40s. = £2.$$

ON SIMPLE EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

First Method.

(p. 68.)—Ex. 3. $5x+3y=38$ .... (1)

$4x - y = 10$ .... (2)

From (1) $y = \frac{38-5x}{3}$

...... (2) $y = 4x - 10$ .... (α).

(p. 70.)—Ex. 7.
$$5x-4y=19 \quad \ldots\ldots \quad (1)$$
$$4x+2y=36 \quad \ldots\ldots \quad (2)$$

Mult. (2) by 2, and $8x+4y=72$
And equation (1) $5x-4y=19$

$\therefore$ by addition, $13x=91$
and $x=7$.

From (2) we get by transposition,
$$2y=36-4x,$$
$$=36-28,$$
$$=8;$$
$$\therefore y=4.$$

Ex. 8.
$$3x+7y=79 \quad \ldots\ldots \quad (1)$$
$$2y-\frac{1}{2}x=9 \quad \ldots\ldots \quad (2)$$

Mult. (2) by 6, then $12y-3x=54$ ...... (α)
Add this equation to (1) and $19y=133$,
$\therefore y=7$.

By dividing equation (α) by 3, and transposing, we get
$$x=4y-18$$
$$=28-18 \text{ (since } y=7\text{; and } \therefore 4y=28\text{)},$$
$$=10.$$

Ex. 9.
$$\frac{x+y}{3}+1=6 \quad \ldots\ldots \quad (1)$$
$$\frac{x-y}{7}+3=4 \quad \ldots\ldots \quad (2)$$

Transposing (1), $\frac{x+y}{3}=5$; and $\therefore x+y=15$ (α)

.... (2) $\frac{x-y}{7}=1$; and $\therefore x-y=7$ (β)

By adding (β) to (α), $2x=22$;
$\therefore x=11$.

By subtracting (β) from (α) $2y=8$;
$\therefore y=4$.

(p. 71.)—Ex. 10. $\frac{x+y}{3}-2y=2 \quad \ldots\ldots (1)$

$$\frac{2x-4y}{5}+y=\frac{23}{5} \quad \ldots\ldots (2)$$

Clearing (1) of fractions, $x+y-6y=6$

or $x-5y=6 \quad \ldots (\alpha)$

.... (2) .... $2x-4y+5y=23$

$2x+y=23 \quad \ldots (\beta)$

Mult. (α) by 2, and $2x-10y=12$

$\therefore$ by subtraction, $11y=11$;

and $\therefore y=1$.

And by (α) $x=6+5y=6+5=11$.

Ex. 11. $\frac{2x-3}{2}+y=7 \quad \ldots\ldots (1)$

$$5x-13y=\frac{67}{2} \quad \ldots\ldots (2)$$

From (1) $2x-3+2y=14$

$2x+2y=17$

$$\therefore x=\frac{17}{2}-y \quad \ldots\ldots (\alpha)$$

By substituting (*second method*) this value of x in (2),

$$5\left(\frac{17}{2}-y\right)-13y=\frac{67}{2}$$

$$\frac{85}{2}-5y-13y=\frac{67}{2}$$

$$18y=\frac{85}{2}-\frac{67}{2}=\frac{18}{2}$$

$$\therefore y=\frac{1}{2}$$

And by (α), $x=\frac{17}{2}-\frac{1}{2}=\frac{16}{2}=8$.

(p. 71.)—Ex. 12. $\frac{3x-7y}{3}=\frac{2x+y+1}{5}$ ...... (1)

$8-\frac{x-y}{5}=6$ ...... (2)

From (1) $15x-35y=6x+3y+3$

$9x-38y=3$ .... (α)

From (2) $\frac{x-y}{5}=2$

$x-y=10;$

$\therefore y=x-10.$ .... (β)

Substitute this value of y in (α), and

$$9x-38(x-10)=3$$
$$9x-38x+380=3$$
$$38x-9x=380-3$$
$$29x=377;$$
$$\therefore x=13.$$

And by (β) $y=x-10=13-10=3.$

(p. 72.)—Ex. 2.

$$\left.\begin{aligned} x+y+z&=90 \quad \ldots\ldots (1)\\ 2x+40&=3y+20 \quad \ldots\ldots (2)\\ 2x+40&=4z+10 \quad \ldots\ldots (3)\end{aligned}\right\}$$

From (1) $x=90-y-z.$ ...... (α)

By substituting this value of x in (2), we get

$$2(90-y-z)+40=3y+20$$
$$180-2y-2z+40=3y+20$$
$$5y+2z=200 \quad \ldots\ldots (\beta)$$

And by inserting the same value of x in (3)

$$2(90-y-z)+40=4z+10$$
$$180-2y-2z+40=4z+10$$
$$2y+6z=210$$

Mult. (β) by 3, and $15y+6z=600;$

$\therefore$ by subtraction, $13y=390,$

$\therefore y=30.$

And by (β), $2z=200-5y$

$=200-150$

$=50;$

$\therefore z=25.$

And by (α) we have also, $x = 90 - y - z$
$= 90 - 30 - 25$
$= 35.$

(p. 72.)—Ex. 3.

$$\left.\begin{aligned} x + y + z &= 53 \quad \ldots\ldots (1) \\ x + 2y + 3z &= 105 \quad \ldots\ldots (2) \\ x + 3y + 4z &= 134 \quad \ldots\ldots (3) \end{aligned}\right\}$$

Subtract (1) from (2), and $y + 2z = 52$ ...... (α)
.... (2) from (3) ... $y + z = 29$ ...... (β)
.... (β) from (α) ... $z = 23.$

By (β), $y = 29 - z = 29 - 23 = 6.$

And by (1), $x = 53 - y - z = 53 - 6 - 23 = 24.$

PROBLEMS.

(p. 74.)—Prob. 4. What two numbers are those to one-third the sum of which if I add 13, the result shall be 17; and if from half their difference I subtract *one*, the remainder shall be *two?*

Let x and y be the numbers;
then $x + y =$ the sum of the numbers,
and $x - y =$ the difference of the numbers,

Hence, by the conditions of the problem,

$$\left.\begin{aligned} \frac{x+y}{3} + 13 = 17, \text{ or } \frac{x+y}{3} &= 4 \quad \ldots\ldots (1) \\ \text{and } \frac{x-y}{2} - 1 = 2, \text{ or } \frac{x-y}{2} &= 3 \quad \ldots\ldots (2) \end{aligned}\right\}$$

By clearing (1) and (2) of fractions, they become

$$x + y = 12$$
$$x - y = 6$$

$\therefore$ by addition $2x = 18$
$x = 9$

And by subtraction, $2y = 6$
$\therefore y = 3.$

(p. 74.)—Prob. 5. There is a certain fraction, such, that if I add *one* to its numerator, it becomes $\frac{1}{2}$; if 3 be added to the denominator, it becomes $\frac{1}{3}$. What is the fraction?

Let $\frac{x}{y}$ represent the required fraction;

$$\left.\begin{array}{ll} \text{then } \frac{x+1}{y}=\frac{1}{2}, \text{ or } 2x+2=y & \ldots\ldots \quad (1) \\ \text{and } \frac{x}{y+3}=\frac{1}{3}, \text{ or } 3x=y+3 & \ldots\ldots \quad (2) \end{array}\right\}$$

By transposition, (2) becomes

$$3x-3=y$$

But by (1) $\qquad 2x+2=y$

$\therefore$ by subtraction, $\quad x-5=0$

and $\therefore$ $x=5$.

And by (1) $y=2x+2=10+2=12$.

Hence the required fraction is $\frac{5}{12}$.

Prob. 7. A person has two horses, and a saddle worth £10; if the saddle be put on the back of the *first* horse, the value becomes *double* that of the *second;* but if the saddle be put on the *second* horse, *his* value will not amount to that of the first horse by £13. What is the value of each horse?

Let x be the value of the 1st horse in £
and y .... 2d ....

Then by the problem, $\left.\begin{array}{r} x+10=2y \\ \text{and } y+10=x-13 \end{array}\right\}$

By transposition, these equations become

$$x=2y-10$$
$$\text{and } x=y+23.$$

By equating these two values of x, we get

$$2y-10=y+23;$$

$\therefore y=£33$, the value of the 2d horse,
and $x=2y-10=66-10=£56$ .... 1st ......

(p. 75.)—PROB. 9. There are two numbers, such, that $\frac{1}{2}$ the greater added to $\frac{1}{3}$ the less is 13; and if $\frac{1}{2}$ the less be taken from $\frac{1}{3}$ the greater, the remainder is nothing. What are the numbers?

Let x denote the greater number,
and y .... less ....

Then by the problem, $\frac{x}{2}+\frac{y}{3}=13$

and $\frac{x}{3}-\frac{y}{2}=0$.

By clearing of fractions, the equations become

$$3x+2y=78$$

and $2x-3y=0$.

From the 2[d] we get $y=\frac{2x}{3}$, and putting this value in the 1[st], we obtain

$$3x+2\left(\frac{2x}{3}\right)=78$$

$$9x+4x=234$$

$$13x=234;$$

$$\therefore x=18,$$

and $y=\frac{2x}{3}=12$.

PROB. 10. There is a certain number, to the sum of whose digits if you add 7, the result will be three times the left-hand digit; and if from the number itself you subtract 18, the digits will be *inverted*. What is the number?

Let $x=$ the *left-hand* digit,
and $y=$ the *right-hand* digit;

Then $10x+y=$ the number itself,
and $10y+x=$ the number with its digits *inverted*.

Hence, by the problem, $x+y+7=3x$
and $10x+y-18=10y+x$;

By transposition and reduction these equations become

$$2x-y=7$$
$$\text{and}\quad x-y=2$$

$\therefore$ by subtraction, we get $x=5$

And $y=x-2=3$;

Hence the number $(10x+y)$ is 53.

(p. 75.)—PROB. 11. A merchant has two kinds of tea, one worth 9*s.* 6*d.* per lb.; the other 13*s.* 6*d.* How many lbs. of each must he take to form a chest of 104 lbs., which shall be worth £56?

Let $x=$ the number of lbs. at 9*s.* 6*d.* per lb.
and $y=$ .... 13*s.* 6*d.* ......

Then, by the problem, $x+y=104$ ...... (1)

$19x+27y=2220$ ...... (2)

Mult. (1) by 19, and $19x+19y=1976$

Subtract this eq^n. from (2) and $8y=264$;

$\therefore y=33$ lbs., at 13*s.* 6*d.* ℔ lb.

and $x=104-y=71$ lbs. ... 9*s.* 6*d.* ......

PROB. 12. A vessel containing 120 gallons is filled in 10 minutes by two spouts running *successively*; the one runs 14 gallons in a minute, the other 9 gallons in a minute. For what time has *each* spout run?

Let $x=$ the No. of minutes run by the 1st spout,
and $y=$ .... 2^d ......
$\therefore 14x=$ the No. of gallons run by the 1st ......
and $9y=$ .... 2^d ......

Then, by the problem, $x+y=10$ ...... (1)

and $14x+9y=120$ ...... (2)

Now (1) $\times$14 gives $14x+14y=140$

And by (2) we have $14x+9y=120$

$\therefore$ by subtraction we get $5y=20$;

and $\therefore y=4$ minutes,

and by (1) $x=10-y=6$ ....

(p. 75.)—PROB. 13. To find three numbers, such, that the *first* with $\frac{1}{2}$ the sum of the *second* and *third* shall be 120; the *second* with $\frac{1}{5}^{th}$ the difference of the *third* and *first* shall be 70; and $\frac{1}{2}$ the sum of the three numbers shall be 95.

Let x, y, z, represent the 1st, 2d, and 3d Nos. respectively;

Then, by the problem,

$$\left.\begin{aligned} x+\frac{1}{2}\left(y+z\right) &= 120 \text{ ...... } (1) \\ y+\frac{1}{5}\left(z-x\right) &= 70 \text{ ...... } (2) \\ \frac{1}{2}\left(x+y+z\right) &= 95 \text{ ...... } (3) \end{aligned}\right\}$$

Subtract (3) from (1) and there results, $\frac{1}{2}x = 25$;

and $\therefore x = 50$.

Substitute this value of x in (1) and (2), and after reduction, they become

$$y + z = 140$$

$$\text{and } y + \frac{1}{5}z = 80.$$

And by subtraction, we get

$$\frac{4}{5}z = 60$$

$$\therefore z = 75.$$

And lastly, from (3) we have, after multiplying it by 2,

$$x+y+z = 190$$

$$\therefore y = 190-x-z$$

$$= 190-50-75$$

$$= 65.$$

CHAPTER IV.

ON INVOLUTION AND EVOLUTION.

ON THE INVOLUTION OF COMPOUND ALGEBRAIC QUANTITIES.

(p. 78.)—Ex. 4.

$$\begin{array}{l}
a+3b \\
a+3b \\
\hline
a^2+3ab \\
\quad +3ab+9b^2 \\
\hline
a^2+6ab+9b^2 = \text{Square of } (a+3b) \\
a+3b \\
\hline
a^3+6a^2b+9ab^2 \\
\quad +3a^2b+18ab^2+27b^3 \\
\hline
a^3+9a^2b+27ab^2+27b^3 = \text{Cube of } (a+3b) \\
a+3b \\
\hline
a^4+9a^3b+27a^2b^2+27ab^3 \\
\quad +3a^3b+27a^2b^2+81ab^3+81b^4 \\
\hline
a^4+12a^3b+54a^2b^2+108ab^3+81b^4 = 4^{\text{th}} \text{ power of } (a+3b)
\end{array}$$

Ex. 5.

$$\begin{array}{l}
3x^2+2x+5 \\
3x^2+2x+5 \\
\hline
9x^4+6x^3+15x^2 \\
\quad +6x^3+4x^2+10x \\
\quad\quad +15x^2+10x+25 \\
\hline
9x^4+12x^3+34x^2+20x+25.
\end{array}$$

(p. 78.)—Ex. 6.

$$\begin{array}{l}
3x - \;\; 5 \\
3x - \;\; 5 \\
\hline
9x^2 - 15x \\
\quad - 15x + 25 \\
\hline
9x^2 - 30x + 25 = \text{Square of } (3x-5) \\
3x - \;\; 5 \\
\hline
27x^3 - 90x^2 + 75x \\
\quad - 45x^2 + 150x - 125 \\
\hline
27x^3 - 135x^2 + 225x - 125 = \text{Cube of } (3x-5).
\end{array}$$

Ex. 7.

$$\begin{array}{l}
x^2 - 2x + 1 \\
x^2 - 2x + 1 \\
\hline
x^4 - 2x^3 + x^2 \\
\quad - 2x^3 + 4x^2 - 2x \\
\qquad + x^2 - 2x + 1 \\
\hline
x^4 - 4x^3 + 6x^2 - 4x + 1 \\
x^2 - 2x + 1 \\
\hline
x^6 - 4x^5 + 6x^4 - 4x^3 + x^2 \\
\quad - 2x^5 + 8x^4 - 12x^3 + 8x^2 - 2x \\
\qquad x^4 - 4x^3 + 6x^2 - 4x + 1 \\
\hline
x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1
\end{array}$$

Ex. 8.

$$\begin{array}{l}
a + b + c \\
a + b + c \\
\hline
a^2 + ab + ac \\
\quad + ab + b^2 + bc \\
\qquad + ac + bc + c^2 \\
\hline
a^2 + 2ab + b^2 + 2ac + 2bc + c^2 = \text{Square of } (a+b+c).
\end{array}$$

ON THE EVOLUTION OF ALGEBRAIC QUANTITIES.

(p. 80.)—Ex. 3.

$$\begin{array}{r|l} & 4x^2+4xy+y^2\,(2x+y \\ & 4x^2 \\ \hline 4x+y) & 4xy+y^2 \\ & 4xy+y^2 \\ \hline & * \quad * \end{array}$$

Ex. 4.

$$\begin{array}{r|l} & 25a^2+30ab+9b^2\,(5a+3b \\ & 25a^2 \\ \hline 10a+3b) & +30ab+9b^2 \\ & +30ab+9b^2 \\ \hline & * \quad * \end{array}$$

Ex. 5.

$$\begin{array}{r|l} & 9x^4+12x^3+22x^2+12x+9\,(3x^2+2x+3 \\ & 9x^4 \\ \hline 6x^2+2x) & +12x^3+22x^2 \\ & +12x^3+\ 4x^2 \\ \hline 6x^2+4x+3) & 18x^2+12x+9 \\ & 18x^2+12x+9 \\ \hline & * \quad * \quad * \end{array}$$

(p. 81.)—Ex. 6.

$$
\begin{array}{l}
4x^4-16x^3+24x^2-16x+4\left(2x^2-4x+2\right. \\
4x^4 \\
\hline
4x^2-4x)\ -16x^3+24x^2 \\
\qquad -16x^3+16x^2 \\
\hline
4x^2-8x+2)\ 8x^2-16x+4 \\
\qquad 8x^2-16x+4 \\
\hline
\quad * \quad * \quad *
\end{array}
$$

Ex. 7.

$$
\begin{array}{l}
36x^4-36x^3+17x^2-4x+\frac{4}{9}\left(6x^2-3x+\frac{2}{3}\right. \\
36x^4 \\
\hline
12x^2-3x)\ -36x^3+17x^2 \\
\qquad -36x^3+9x^2 \\
\hline
12x^2-6x+\frac{2}{3})\ 8x^2-4x+\frac{4}{9} \\
\qquad 8x^2-4x+\frac{4}{9} \\
\hline
\quad * \quad * \quad *
\end{array}
$$

Ex. 8.

$$
\begin{array}{l}
x^4+8x^2+24+\frac{32}{x^2}+\frac{16}{x^4}\left(x^2+4+\frac{4}{x^2}\right. \\
x^4 \\
\hline
2x^2+4)\ +8x^2+24 \\
\qquad +8x^2+16 \\
\hline
2x^2+8+\frac{4}{x^2})\ 8+\frac{32}{x^2}+\frac{16}{x^4} \\
\qquad 8+\frac{32}{x^2}+\frac{16}{x^4} \\
\hline
\quad * \quad * \quad *
\end{array}
$$

ON THE EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

(p. 83.)—Ex. 1.

```
      105625 (325
      9
     ----
  62) 156
      124
     -----
 645) 3225
      3225
     -----
        *
```

Ex. 2.

```
      173056 (416
      16
     ----
  81) 130
       81
     -----
 826) 4956
      4956
     -----
        *
```

Ex. 3.

```
       5934096 (2436
       4
      ----
   44) 193
       176
      -----
  483) 1740
       1449
      ------
 4866) 29196
       29196
      ------
          *
```

CHAP. V. QUADRATIC EQUATIONS.

ON THE SOLUTION OF PURE QUADRATIC EQUATIONS.

(p. 84.)—Ex. 4. $5x^2-1=244.$

By transposition, $5x^2=244+1$
$5x^2=245$
$x^2=49;$

Extracting the sq. root, $x=\pm 7.$

Ex. 5. $9x^2+9=3x^2+63$

By transposition, $9x^2-3x^2=63-9$
$6x^2=54$
$x^2=9;$

∴ by extracting sq. root, $x=\pm 3.$

Ex. 6. $\frac{4x^2+5}{9}=45.$

$4x^2+5=405$
$4x^2=400$
$x^2=100;$
$\therefore x=\pm 10.$

Ex. 7. $bx^2+c+3=2bx^2+1.$

By transposition, $2bx^2-bx^2=c+3-1$
$bx^2=c+2$

By dividing by b, $x^2=\frac{c+2}{b}$

and ∴ extracting the sq. root, $x=\pm\sqrt{\frac{c+2}{b}}$

ON THE SOLUTION OF ADFECTED QUADRATIC EQUATIONS.

(p. 86.)—Ex. 3. $x^2+12x=108.$

By adding 6^2 or 36 to each side, the equation becomes
$x^2+12x+36=144$

Extracting the sq. root, $x+6=\pm 12$

and $\therefore x=+12-6=6.$

or $x=-12-6=-18.$

(p. 86.)—Ex. 4. $x^2-14x=51.$

Add 7^2 or 49 to each side, and

$$x^2-14x+7^2=51+49=100;$$

By extracting the sq. root, $x-7=\pm 10$;

$$\therefore x=\pm 10+7=17 \text{ or } -3.$$

Ex. 5. $x^2-8x=20$

Add 4^2 or 16 to each side, and

$$x^2-8x+4^2=20+16=36$$

$$x-4=\pm 6;$$

$$\therefore x=10 \text{ or } -2.$$

(p. 87.)—Ex. 8. $x^2+7x=78.$

Adding $\left(\frac{7}{2}\right)^2$ or $\frac{49}{4}$ to each side, the equation becomes

$$x^2+7x+\left(\frac{7}{2}\right)^2=78+\frac{49}{4}=\frac{312+49}{4}=\frac{361}{4}$$

Extracting the sq. root, $x+\frac{7}{2}=\pm\frac{19}{2}$;

$$\therefore x=\pm\frac{19}{2}-\frac{7}{2}=6 \text{ or } -13$$

Ex. 9. $x^2+3x=28.$

To each side add $\left(\frac{3}{2}\right)^2$ or $\frac{9}{4}$, and we have

$$x^2+3x+\left(\frac{3}{2}\right)^2=28+\frac{9}{4}=\frac{112+9}{4}=\frac{121}{4}$$

By extracting the sq. root, $x+\frac{3}{2}=\pm\frac{11}{2}$

$$\therefore x=\pm\frac{11}{2}-\frac{3}{2}=4 \text{ or } -7.$$

(p. 87.)—Ex. 10. $x^2-3x=40.$

Compl^g. the sq., $x^2-3x+\left(\frac{3}{2}\right)^2=40+\frac{9}{4}=\frac{160}{4}+\frac{9}{4}=\frac{169}{4}$

Extracting the sq. root, $x-\frac{3}{2}=\pm\frac{13}{2}$

$$\therefore x=8 \text{ or} -5.$$

Ex. 11. $x^2+x=30.$

Here the *coefficient* of x^2 is 1; by adding therefore $\left(\frac{1}{2}\right)^2$ or $\frac{1}{4}$ to each side

$$x^2+x+\left(\frac{1}{2}\right)^2=30+\frac{1}{4}=\frac{121}{4}$$

$$x+\frac{1}{2}=\pm\frac{11}{2}$$

$$\therefore x=5 \text{ or } -6.$$

Ex. 14. $3x^2+2x=161.$

Dividing equ^n. by 3, $x^2+\frac{2}{3}x=\frac{161}{3}$

Compl^g. the sq. $x^2+\frac{2}{3}x+\left(\frac{1}{3}\right)^2=\frac{161}{3}+\frac{1}{9}=\frac{483}{9}+\frac{1}{9}=\frac{484}{9}$

$$x+\frac{1}{3}=\pm\frac{22}{3}$$

$$\therefore x=7 \text{ or } -7\frac{2}{3}$$

(p. 88.)—Ex. 15. $2x^2-5x=117.$

Divid^g. each term by 2, $x^2-\frac{5}{2}x=\frac{117}{2}$

$$x^2-\frac{5}{2}x+\left(\frac{5}{4}\right)^2=\frac{117}{2}+\frac{25}{16}=\frac{936}{10}+\frac{25}{16}=\frac{961}{16}$$

$$x-\frac{5}{4}=\pm\frac{31}{4}$$

$$\therefore x=9 \text{ or } -\frac{13}{2}.$$

(p. 88.)—Ex. 16. $3x^2-2x=280.$

$$x^2-\frac{2}{3}x=\frac{280}{3}$$

$$x^2-\frac{2}{3}x+\left(\frac{1}{3}\right)^2=\frac{280}{3}+\frac{1}{9}=\frac{840}{9}+\frac{1}{9}=\frac{841}{9}$$

$$x-\frac{1}{3}=\pm\frac{29}{3}$$

$$\therefore x=10 \text{ or } -9\frac{1}{3}.$$

Ex. 17. $4x^2-7x=492.$

$$x^2-\frac{7}{4}x=\frac{492}{4}$$

$$x^2-\frac{7}{4}x+\left(\frac{7}{8}\right)^2=\frac{492}{4}+\frac{49}{64}=\frac{7921}{64}$$

$$x-\frac{7}{8}=\pm\frac{89}{8}$$

$$\therefore x=12 \text{ or } -10\frac{1}{4}.$$

(p. 89.)—Ex. 3. $\frac{x^2}{6}-1=x+11.$

$$x^2-6=6x+66$$
$$x^2-6x=72$$
$$x^2-6x+9=81$$
$$x-3=\pm 9;$$
$$\therefore x=12 \text{ or } -6.$$

Ex. 4. $\frac{2x}{3}+\frac{1}{x}=\frac{7}{3}.$

Mult. by $3x$, and $2x^2+3=7x$

$$2x^2-7x=-3.$$

$$x^2-\frac{7}{2}x=-\frac{3}{2}$$

$$x^2-\frac{7}{2}x+\left(\frac{7}{4}\right)^2=\frac{49}{16}-\frac{3}{2}=\frac{25}{16}$$

$$x-\frac{7}{4}=\pm\frac{5}{4};$$

$$\therefore x=3 \text{ or } \frac{1}{2}.$$

(p. 89.)—Ex. 5. $$\frac{x^2}{3}-\frac{x}{2}=9.$$

Mult. by 3, and $x^2-\frac{3}{2}x=27$

Complete the sq. and $x^2-\frac{3}{2}x+\left(\frac{3}{4}\right)^2=27+\frac{9}{16}=\frac{441}{16}$

$$x-\frac{3}{4}=\pm\frac{21}{4};$$

$$\therefore x=6 \text{ or } -\frac{9}{2}.$$

Ex. 6. $$\frac{6}{x+1}+\frac{2}{x}=3$$

Mult. by $x+1$, and $6+\frac{2x+2}{x}=3x+3$

.... x, ... $6x+2x+2=3x^2+3x$

$$8x+2=3x^2+3x$$

By transposition, $3x^2-5x=2$

Dividing by 3, $x^2-\frac{5}{3}x=\frac{2}{3}$

By completing the sq. $x^2-\frac{5}{3}x+\left(\frac{5}{6}\right)^2=\frac{2}{3}+\frac{25}{36}=\frac{49}{36}$

$$x-\frac{5}{6}=\pm\frac{7}{6};$$

$$\therefore x=2 \text{ or } -\frac{1}{3}.$$

Ex. 7. $$x^2-34=\frac{1}{3}x.$$

$$x^2-\frac{1}{3}x=34$$

$$x^2-\frac{1}{3}x+\left(\frac{1}{6}\right)^2=34+\frac{1}{36}=\frac{1225}{36}$$

$$x-\frac{1}{6}=\pm\frac{35}{6}$$

$$\therefore x=6 \text{ or } -5\frac{2}{3}.$$

(p. 89.)—Ex. 8. $\frac{x}{5}+\frac{5}{x}=5\frac{1}{5}$

Mult. by $5x$, and $x^2+25=26x$

$$x^2-26x=-25$$

$$x^2-26x+(13)^2=169-25=144$$

$$x-13=\pm 12;$$

$$\therefore x=25 \text{ or } 1.$$

Ex. 9. $x+\frac{24}{x-1}=3x-4$

By transposition, $\frac{24}{x-1}=2x-4$

Multiplying by $x-1$, $24=2x^2-6x+4.$

By transposing again, $2x^2-6x=20$

$$x^2-3x=10$$

$$x^2-3x+\left(\frac{3}{2}\right)^2=10+\frac{9}{4}=\frac{49}{4}$$

$$x-\frac{3}{2}=\pm\frac{7}{2}$$

$$\therefore x=5 \text{ or } -2.$$

Ex. 10. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}.$

$$2x^2+2x+1=\frac{13(x^2+x)}{6}$$

$$12x^2+12x+6=13x^2+13x$$

$$x^2+x=6$$

$$x^2+x+\left(\frac{1}{2}\right)^2=6+\frac{1}{4}=\frac{25}{4}$$

$$x+\frac{1}{2}=\pm\frac{5}{2};$$

$$\because x=2 \text{ or } -3.$$

(p. 89.)—Ex. 11. $\frac{3x}{x+2}-\frac{x-1}{6}=x-9.$

Mult. by 6, and $\frac{18x}{x+2}-x+1=6x-54$

Transposing, $\frac{18x}{x+2}=7x-55$

Mult. by $x+2$, and $18x=7x^2-41x-110.$

$$7x^2-59x=110.$$

$$x^2-\frac{59}{7}x=\frac{110}{7}$$

$$x^2-\frac{59}{7}x+\left(\frac{59}{14}\right)^2=\frac{110}{7}+\frac{3481}{196}=\frac{6561}{196}$$

$$x-\frac{59}{14}=\pm\frac{81}{14}$$

$$\therefore x=\pm\frac{81}{14}+\frac{59}{14}=10 \text{ or } -\frac{11}{7}.$$

(p. 90.)—Ex. 13. $x^2-2x=-2.$

$$x^2-2x+1=1-2=-1.$$

$$x-1=\pm\sqrt{-1}$$

$$\therefore x=1\pm\sqrt{-1}$$

Ex. 14. $x^2-16x=-15.$

$$x^2-16x+8^2=64-15=49$$

$$x-8=\pm 7;$$

$$\therefore x=15 \text{ or } 1.$$

Ex. 16. $x^2-6x+19=13.$

$$x^2-6x=-6$$

$$x^2-6x+9=9-3=3$$

$$x-3=\pm\sqrt{3}=\pm 1.732;$$

$$\therefore x=4.732 \text{ or } 1.268.$$

(p. 90.)—Ex. 17. $5x^2+4x=25.$

$$x^2+\frac{4}{5}x=\frac{25}{5}$$

$$x^2+\frac{4}{5}x+\left(\frac{2}{5}\right)^2=\frac{25}{5}+\frac{4}{25}=\frac{129}{25}$$

$$x+\frac{2}{5}=\frac{\pm\sqrt{129}}{5}=\pm\frac{11.357}{5}$$

$$\therefore x=\frac{9.357}{5} \text{ or } -\frac{13.357}{5}$$

$$\text{or } x=1.871 \text{ or } -2.671.$$

(p. 91.) Ex. 20. $x^4+4x^2=12.$

$$x^4+4x^2+4=16$$

$$x^2+2=\pm4.$$

$$x^2=2 \text{ or } -6.$$

$$\therefore x=\pm\sqrt{2} \text{ or } \pm\sqrt{-6}.$$

Ex. 21. $x^6-8x^3=513.$

$$x^6-8x^3+16=529$$

$$x^3-4=\pm23$$

$$x^3=27 \text{ or } -19$$

$$\therefore x=3 \text{ or } \sqrt[3]{-19}.$$

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

(p. 96.)—Prob. 7. To divide the number 33 into two such parts that their product shall be 162.

Let $x=$ one part,
then $33-x=$ the other part,
and $x(33-x)=$ *the product* of the two parts.

Hence, by the problem, $x(33-x)=162$,
or $33x-x^2=162$.

By transposition, $x^2-33x=-162$.

Complet^g. the sq. $x^2-33x+\left(\frac{33}{2}\right)^2=\frac{1089}{4}-162=\frac{441}{4}$

$$x-\frac{33}{2}=\pm\frac{21}{2}$$

$$\therefore x=\frac{33}{2}\pm\frac{21}{2}=27 \text{ or } 6,$$

$$\text{and } 33-x=6 \text{ or } 27.$$

Hence it appears that the *answers* are the same, but in an inverted order.

(p. 96.)—Prob. 8. What two numbers are those whose sum is 29, and product 100?

Let $x=$ one number;

then $29-x=$ the other number,

and $x(29-x)=$ *the product* of the numbers.

Hence, by the problem, $x(29-x)=100$

$$29x-x^2=100$$

$$x^2-29x=-100$$

$$x^2-29x+\left(\frac{29}{2}\right)^2=\frac{841}{4}-100=\frac{441}{4}$$

$$x-\frac{29}{2}=\frac{21}{2}$$

$\therefore x=35$, the one part,

and $29-x=4$, the other.

Prob. 9. The difference of two numbers is 5, and $\frac{1}{4}$th part of their product is 26. What are the numbers?

Let $x=$ the less number,

then $x+5=$ the greater number,

and $x(x+5)=$ *the product* of the two numbers;

$\therefore \frac{x(x+5)}{4}=\frac{1}{4}$th of the product.

But by the problem, $\frac{1}{4}$th of the product $=26$;

$$\therefore \frac{x(x+5)}{4}=26$$

$$x^2+5x=104$$

$$x^2+5x+\left(\frac{5}{2}\right)^2=104+\frac{25}{4}=\frac{441}{4}$$

$$x+\frac{5}{2}=\frac{21}{2}$$

$$\therefore\ x=8, \text{ the less number,}$$

$$\text{and } x+5=13, \text{ the greater number.}$$

(p. 96.)—PROB. 10. The difference of two numbers is 6; and if 47 be added to *twice the square of the less*, it will be equal to the *square of the greater*. What are the numbers?

Let x = the less number,
then $x+6$ = the greater number.

Hence, by the problem, $2x^2+47=(x+6)^2$

$$2x^2+47=x^2+12x+36$$

$$x^2-12x=-11.$$

$$x^2-12x+36=36-11=25$$

$$x-6=5;$$

$$\therefore x=11, \text{ the less number,}$$

$$\text{and } x+6=17, \text{ the greater nnmber.}$$

PROB. 11. There are two numbers whose sum is 30; and $\frac{1}{3}$d of their product *plus* 18 is equal to the square of the *less* number. What are the numbers?

Let x = the less number,
then $30-x$ = the greater number,

and $\frac{x(30-x)}{3}$ = $\frac{1}{3}$d of their product.

Hence, by the problem, $\frac{x(30-x)}{3}+18=x^2$

$$30x-x^2+54=3x^2$$

$$4x^2-30x=54$$

$$x^2-\frac{30}{4}x=\frac{54}{4}$$

$$x^2-\frac{30}{4}x+\left(\frac{15}{4}\right)^2=\frac{54}{4}+\frac{225}{16}=\frac{441}{16}$$

$$x-\frac{15}{4}=\frac{21}{4}$$

$$\therefore\ x=9, \text{ the less No.}$$

$$\text{and } 30-x=21, \text{ the greater No.}$$

(p. 96.)—PROB. 12. There are two numbers whose product is 120. If 2 be added to the less and 3 subtracted from the greater, the product of the sum and remainder will also be 120. What are the numbers?

Let $x=$ the greater number,

then $\frac{120}{x}=$ the less number.

Hence, by the conditions of the problem, we get the equation

$$\left(\frac{120}{x}+2\right)\times(x-3)=120.$$

$$120+2x-\frac{360}{x}-6=120.$$

Or, by reduction, $x^2-3x=180$

$$x^2-3x+\left(\frac{3}{2}\right)^2=180+\frac{9}{4}=\frac{724}{4}$$

$$x-\frac{3}{2}=\frac{27}{2}$$

$\therefore x=15$, the greater number,

and $\frac{120}{x}=8$, the less number.

PROB. 13. A and B distribute £1200 each among a certain number of persons: A relieves 40 persons more than B, and B gives £5 a-piece to each person *more* than A. How many persons were relieved by A and B respectively?

Let $x=$ the number of persons relieved by B,

then $x+40=$ .... A

$\therefore \frac{1200}{x}=$ the No. of £ B gives to each person,

and $\frac{1200}{x+40}=$ .... A ....

But B gives £5 more to each person than A gives;

$$\therefore \frac{1200}{x+40}+5=\frac{1200}{x}$$

By clearing this equation of fractions, it becomes

$$1200x+5x^2+200x=1200x+48000$$

Or, by reduction, $x^2+40x=9600$

$$x^2+40x+(20)^2=10000$$
$$x+20=100;$$
$$\therefore x=80, \text{ the No. relieved by B,}$$
$$\text{and } x+40=120, \ldots\ldots\ldots\ldots \text{ A.}$$

(p. 96.)—PROB. 14. A person bought a certain number of sheep for £120. If there had been 8 more, each sheep would have cost him 10 shillings less. How many sheep were there ?

Let $x=$ the No. of sheep,

then $x+8=$ .... if there had been 8 more ;

$\therefore \frac{120}{x}=$ the price in £ of each sheep,

and $\frac{120}{x+8}=$ .... if there had been 8 more.

But if there had been 8 more, each sheep would have cost 10*s*. or $\frac{1}{2}$£ less.

$$\therefore \frac{120}{x}-\frac{1}{2}=\frac{120}{x+8}.$$

This equation when reduced becomes

$$x^2+8x=1920$$
$$x^2+8x+16=1936$$
$$x+4=44;$$
$$\therefore x=40.$$

(p. 97.)—PROB. 16. A and B set off at the same time to a place at the distance of 300 miles. A travels at the rate of one mile an hour faster than B, and arrives at his journey's end 10 hours before him. At what rate did each person travel per *hour ?*

Let $x=$ the number of miles B travels per hour,

then $x+1=$ .... A ....

$\therefore \frac{300}{x}=$ the No. of hours B takes to travel the distance.

and $\frac{300}{x+1}=$ .... A ....

But A arrives at his journey's end 10 hours before B;

$$\therefore \frac{300}{x+1}+10=\frac{300}{x}.$$

This equation becomes by reduction

$$x^2+x=30$$

$$x^2+x+\left(\frac{1}{2}\right)^2=30+\frac{1}{4}=\frac{121}{4}.$$

$$x+\frac{1}{2}=\pm\frac{11}{2}.$$

$\therefore\ x=5$, miles travelled by B ℔ hour.

and $x+1=6$, ... A ....

(p. 97.)—PROB. 18. To divide the number 20 into two such parts, that their product shall be 105.

Let $x=$ one part,

then $20-x=$ the other part;

Hence, $x(20-x)=105$,

or $20x-x^2=105$.

Transpose, and $x^2-20x=-105$.

Complete the equation, and there results

$$x^2-20x+100=-5$$

$$x-10=\pm\sqrt{-5}.$$

$$\therefore x=10\pm\sqrt{-5}.$$

(p. 98).—PROB. 20. To resolve the number 18 into two such factors, that the sum of their cubes shall be 243.

Let x be one factor,

then $\frac{18}{x}$ is the other.

Hence $x^3+\left(\frac{18}{x}\right)^3=243$

$$x^6+5832=243x^3$$

or $x^6-243x^3=-5832$

$$x^6-243x^3+\left(\frac{243}{2}\right)^2=\frac{59049}{4}-5832=\frac{35721}{4}$$

$$x^3-\frac{243}{2}=\pm\frac{189}{2}$$

$$x^3=216 \text{ or } 27\,;$$

$$\therefore x=6 \text{ or } 3,$$

$$\text{and } \frac{18}{x}=3 \text{ or } 6.$$

ON THE SOLUTION OF QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

Case I.—(p. 100.)

Ex. 4. Let $\left.\begin{aligned}2x-3y&=1\\2x^2+xy-5y^2&=20\end{aligned}\right\}$ to find the values of x and y.

From the 1st equation, $2x=1+3y$

$$\therefore x=\frac{1+3y}{2}$$

Putting this value of x in the 2d equation

$$2\left(\frac{1+3y}{2}\right)^2+\left(\frac{1+3y}{2}\right)y-5y^2=20,$$

$$\frac{1+6y+9y^2}{2}+\frac{y+3y^2}{2}-5y^2=20.$$

$$\text{or } y^2+\frac{7}{2}y=\frac{39}{2}$$

$$y^2+\frac{7}{2}y+\left(\frac{7}{4}\right)^2=\frac{39}{2}+\frac{49}{16}=\frac{361}{16}$$

$$y+\frac{7}{4}=\frac{19}{4}$$

$$\therefore y=3,$$

$$\text{and } x=\frac{1+3y}{2}=5.$$

Ex. 5. There are two numbers such, that if the less be taken from three times the greater, the remainder will be 35; and if four times the greater be divided by three times the less *plus* one, the quotient will be equal to the less number. What are the numbers?

Let $x=$ the less number,
and $y=$ the greater number;

Then the conditions of the problem furnish the equations

$$\left.\begin{aligned} 3y-x&=35 \quad \ldots\ldots \quad (1) \\ \frac{4y}{3x+1}&=x \quad \ldots\ldots \quad (2) \end{aligned}\right\}$$

From (1) we get $3y=35+x$;

$$\therefore y=\frac{35+x}{3}.$$

By putting this value in (2), it becomes

$$\frac{4\,(35+x)}{3\,(3x+1)}=x;$$

and by reduction,

$$140+4x=9x^2+3x$$

$$\text{or } x^2-\frac{1}{9}x=\frac{140}{9}$$

$$x^2-\frac{1}{9}x+\left(\frac{1}{18}\right)^2=\frac{140}{9}+\frac{1}{324}=\frac{5041}{324}$$

$$x-\frac{1}{18}=\frac{71}{18}$$

$$x=\frac{71}{18}+\frac{1}{18}=4,$$

$$\text{and } y=\frac{35+x}{3}=\frac{35+4}{3}=13.$$

(p. 100.)—Ex. 6. What number is that, the *sum* of whose digits is 15, and if 31 be added to their *product* the digits will be *inverted?*

Let $x=$ the *left-hand* digit,
and $y=$ the *right-hand* digit,
then $10y+x=$ the No. represented by the digits *inverted*:

Hence the question furnishes the equations

$$\left.\begin{aligned} x+y&=15 \\ xy+31&=10y+x \end{aligned}\right\}$$

From the 1st equation, we get $x=15-y$

$\therefore$ by substitution, $(15-y)\,y+31=10y+15-y$

or, by reduction, $y^2-6y=16$

$$y^2-6y+9=16+9=25$$
$$y-3=5;$$
$$\therefore y=8,$$
$$\text{and } x=15-y=7:$$

Hence the required number is 78.

Case II.—(p. 102.)

Ex. 3. Find two numbers such, that the square of the greater *minus* the square of the less may be 56; and the square of the less *plus* $\frac{1}{3}^{d}$ their product may be 40.

Let $x=$ the greater number,
and $y=$ ... less ....

Then, by the question, we have the equations

$$x^2-y^2=56$$
$$y^2+\frac{1}{3}xy=40.$$

Assume $x=vy$, and the equations become, by substitution,

$$v^2y^2-y^2=56,\text{ or } y^2=\frac{56}{v^2-1}$$
$$y^2+\frac{1}{3}vy^2=40,\text{ or } y^2=\frac{40}{1+\frac{1}{3}v}$$

Hence, we get by equating these two values of y^2,

$$\frac{40}{1+\frac{1}{3}v}=\frac{56}{v^2-1}$$
$$40v^2-40=56+\frac{56}{3}v$$
$$v^2-\frac{7}{15}v=\frac{12}{5}$$
$$v^2-\frac{7}{15}v+\left(\frac{7}{30}\right)^2=\frac{12}{5}+\frac{49}{900}=\frac{2209}{900}.$$

$$v-\frac{7}{30}=\frac{47}{30}$$

$$\therefore v=\frac{54}{30}=\frac{9}{5}.$$

And, since $y^2=\dfrac{40}{1+\frac{1}{3}v}=\dfrac{40}{1+\frac{3}{5}}=25,$

$$y=5,$$

$$\text{and } x=vy=\frac{9}{5}\times 5=9.$$

(p. 102.)—Ex. 4. There are two numbers such, that 3 times the square of the greater plus twice the square of the less is 110; and half their product *plus* the square of the less is 4. What are the numbers?

Let $x=$ the greater number,
and $y=$ the less ....
Then $3x^2+2y^2=110$

$$\frac{1}{2}xy+y^2=4.$$

Assume $x=vy$, and the equations then become

$$\left.\begin{array}{l} 3v^2y^2+2y^2=110, \text{ or } y^2=\dfrac{110}{3v^2+2} \\ \dfrac{1}{2}vy^2+y^2=4, \text{ or } y^2=\dfrac{8}{v+2} \end{array}\right\}$$

Hence $\dfrac{110}{3v^2+2}=\dfrac{8}{v+2};$

or, by reduction, $v^2-\dfrac{55}{12}v=\dfrac{102}{12}.$

$$v^2-\frac{55}{12}v+\left(\frac{55}{24}\right)^2=\frac{102}{12}+\frac{3025}{576}=\frac{7921}{576}$$

$$v-\frac{55}{24}=\frac{89}{24}$$

$$v=6.$$

And, since $y^2=\dfrac{8}{v+2}=\dfrac{8}{6+2}=1$

$$y=1,$$

$$\text{and } x=vy=6.$$

CHAPTER VI.

ON ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSIONS.

On Arithmetical Progression.

(p. 103.)—Ex. 3. Find the 25[th] term of the series, 5, 8, 11, &c.

$$\text{Here } \left.\begin{aligned} a &= 5 \\ d &= 3 \\ n &= 25 \end{aligned}\right\} \quad \begin{aligned} \therefore l &= 5+(25-1)3 \\ &= 5+24\times 3 \\ &= 77. \end{aligned}$$

Ex. 4. Find the 12[th] term of the series 15, 12, 9, &c.

$$\text{Here } \left.\begin{aligned} a &= 15 \\ d &= -3 \\ n &= 12 \end{aligned}\right\} \quad \begin{aligned} \therefore l &= 15+(12-1)\times -3 \\ &= 15-11\times 3 \\ &= -18. \end{aligned}$$

(p. 104.)—Ex. 6. Find 7 arithmetic *means* between 3 and 59.

$$\text{Here } \left.\begin{aligned} a &= 3 \\ l &= 59 \\ n &= 9 \end{aligned}\right\} \quad \begin{aligned} \text{But } a+(n-1)d &= l \\ \therefore 3+8d &= 59 \\ \therefore d &= 7. \end{aligned}$$

And the means required are 10, 17, 24, 31, 38, 45, 52.

Ex. 7. Find 8 arithmetic means between 4 and 67.

$$\left.\begin{aligned} a &= 4 \\ l &= 67 \\ n &= 10 \end{aligned}\right\} \quad \begin{aligned} \therefore 4+(10-1)d &= 67 \\ 4+9d &= 67 \\ \therefore d &= 7. \end{aligned}$$

Hence the means are 11, 18, 25, 32, 39, 46, 53, 60.

Ex. 8. Insert 9 arithmetic means between 9 and 109.

$$\left.\begin{aligned} a &= 9 \\ l &= 109 \\ n &= 11 \end{aligned}\right\} \quad \begin{aligned} \therefore 9+(11-1)d &= 109 \\ 9+10d &= 109 \\ \therefore d &= 10. \end{aligned}$$

Hence the means can be found as in the preceding examples.

To find the sum of an arithmetic series.

(p. 104.)—Ex. 3. Find the sum of 25 terms of the series 2, 5, 8, 11, &c.

Here $a = 2$, $d = 3$, $n = 25$

$$\therefore S = \left\{2a + (n-1)d\right\}\frac{n}{2}$$
$$= \left\{2\times2+(25-1)\times3\right\}\times\frac{25}{2}$$
$$= (2\times2+24\times3)\times\frac{25}{2}$$
$$= (2+36)\times25$$
$$= 38\times25 = 950.$$

Ex. 4. Find the sum of 36 terms of the series 40, 38, 36, 34, &c.

$a = 40$, $d = -2$, $n = 36$

$$\therefore S = \left\{2a + (n-1)d\right\}\frac{n}{2}$$
$$= \left\{2\times40+(36-1)\times-2\right\}\times\frac{36}{2}$$
$$= (80-70)\times18$$
$$= 10\times18$$
$$= 180.$$

(p. 105.)—Ex. 6. Find the sum of 32 terms of the series 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, &c.

$a = 1$, $d = \frac{1}{2}$, $n = 32$

$$\therefore S = \left\{2a + (n-1)d\right\}\frac{n}{2}$$
$$= \left\{2\times1+(32-1)\times\frac{1}{2}\right\}\frac{32}{2}$$
$$= \left(2+\frac{31}{2}\right)16$$
$$= \frac{35}{2}\times16$$
$$= 280.$$

PROBLEMS.

(p. 106.)—Prob. 4. The *sum* of an arithmetic series is 950, the *common difference* 3, and the *number of terms* 25. What is the *first term?*

$$\left.\begin{aligned} \text{Here } S &= 950 \\ d &= 3 \\ n &= 25 \end{aligned}\right\} \quad \begin{aligned} &\text{and } \left\{2a+(n-1)d\right\}\frac{n}{2}=S. \\ &\therefore \quad \left\{2a+(25-1)\times 3\right\}\frac{25}{2}=950 \\ &(a+12\times 3)25=950 \end{aligned}$$

Dividing both sides by 25, $a+36=38$

$$\therefore a=2.$$

Prob. 5. The *sum* of an arithmetic series is 165, the *first term* 3, and the *number of terms* 10. What is the *common difference?*

$$\left.\begin{aligned} S &= 165 \\ a &= 3 \\ n &= 10 \end{aligned}\right\} \quad \begin{aligned} &\left\{2a+(n-1)d\right\}\frac{n}{2}=S \\ &\therefore \left\{2\times 3+(10-1)d\right\}\frac{10}{2}=165 \\ &(6+9d)\,5=165. \end{aligned}$$

Dividing by 5, $6+9d=33,$

$$9d=27;$$

$$\therefore d=3.$$

Prob. 6. The *sum* of an arithmetic series is 440, *first term* 3, and *common difference* 2. What is the *number of terms?*

$$\left.\begin{aligned} S &= 440 \\ a &= 3 \\ d &= 2 \end{aligned}\right\} \quad \begin{aligned} &\left\{2a+(n-1)d\right\}\frac{n}{2}=S \\ &\left\{2\times 3+(n-1)2\right\}\frac{n}{2}=440. \end{aligned}$$

$$n^2+2n=440.$$

$$n^2+2n+1=441,$$

$$n+1=21;$$

$$\therefore n=20.$$

(p. 106.)—Prob. 7. The *sum* of an arithmetic series is 54, the *first term* 14, and *common difference* -2. What is the *number of terms?*

$$\left.\begin{array}{l} S=54 \\ a=14 \\ d=-2 \end{array}\right\} \therefore \text{ since } \left\{2a+(n-1)d\right\}\frac{n}{2}=S$$

$$\left\{2\times14+(n-1)\times-2\right\}\frac{n}{2}=54.$$

$$(15-n)n=54.$$

$$\text{or } x^2-15n=-54,$$

$$n^2-15n+\left(\frac{15}{2}\right)^2=\frac{225}{4}-54=\frac{9}{4},$$

$$n-\frac{15}{2}=\pm\frac{3}{2};$$

$$\therefore n=9 \text{ or } 6.$$

(p. 108.)—Prob. 10. A person bought 47 sheep, and gave 1 shilling for the *first* sheep, 3 for the *second*, 5 for the *third*, and so on. What did *all* the sheep cost him?

$$\left.\begin{array}{l} \text{Here } a=1 \\ d=2 \\ n=47 \end{array}\right\} \therefore S=\left\{2a+(n-1)d\right\}\frac{n}{2}$$

$$=\left\{2\times1+(47-1)2\right\}\frac{47}{2}$$

$$=(1+46)\,47.$$

$$=47\times47,$$

$$=2209s.=£110.\ 9s.$$

Prob. 11. A gentleman began the year by giving away a *farthing* the *first* day, a *halfpenny* the *second*, *three farthings* the *third*, and so on. What money had he disposed of in charity at the end of the year?

$$\left.\begin{array}{l} a=1 \\ d=1 \\ n=365 \end{array}\right\} \therefore S=\left\{2a+(n-1)d\right\}\frac{n}{2}$$

$$=\left\{2\times1+(365-1)\times1\right\}\frac{365}{2}$$

$$=(2+364)\,\frac{365}{2}$$

$$=(1+182)365,$$
$$=183\times365,$$
$$=66795 \text{ farth}^{s}.=£69\ 11s.\ 6\tfrac{3}{4}d.$$

(p. 108.)—PROB. 12. A travels *uniformly* at the rate of 6 miles an hour, and sets off upon his journey 3 hours and 20 minutes before B; B follows him at the rate of 5 miles the *first* hour, 6 the *second*, 7 the *third*, and so on. In how many hours will B overtake A?

Let $x=$ the No. of hours B is in overtaking A,

then, $x+3\frac{1}{3}=$ .... A travels before he is overtaken; and therefore

$$\left(x+3\frac{1}{3}\right)6=\text{ the No. of miles A travels,}$$

and $\left\{2\times5+(x-1)\times1\right\}\frac{x}{2}=$ .... B ....

But A and B both travel the same distance: hence, we have the equation

$$\left\{2\times5+(x-1)\times1\right\}\frac{x}{2}=\left(x+3\frac{1}{3}\right)6.$$

$$(10+x-1)\frac{x}{2}=6x+20$$

$$x^2+9x=12x+40$$

$$x^2-3x=40$$

$$x^2-3x+\left(\frac{3}{2}\right)^2=40+\frac{9}{4}=\frac{169}{4}$$

$$x-\frac{3}{2}=\frac{13}{2}$$

$$\therefore\ x=8 \text{ hours.}$$

(p. 109.)—PROB. 14. There is a certain number of quantities in arithmetic progression, whose *first term* is 2, and whose *sum* is equal to 8 times their number; if 7 be added to the *third* term, and that sum be divided by the number of terms, the quotient will be equal to the *common difference.* What are the numbers?

Let $x=$ the number of terms,
and $y=$ the common difference,

Then, *the sum* of the terms $=\left\{2\times2+(x-1)y\right\}\frac{x}{2}$

But *the sum* of the terms $=8$ times the No. of terms $=8x$.

$$\therefore \left\{2\times2+(x-1)y\right\}\frac{x}{2}=8x;$$

or, multiplying by 2, and dividing by x,

$$4+xy-y=16$$
$$\therefore xy=12+y \quad \ldots\ldots \quad (1)$$

Again, since the 3[d] term $=2+2y$, we have the equation

$$\frac{2+2y+7}{x}=y;$$
$$\therefore xy=9+2y \quad \ldots\ldots \quad (2)$$

Equating the values of xy obtained in (1) and (2), there results

$$2y+9=12+y$$
$$\therefore y=3.$$

And from (1), $x=\frac{12+y}{y}=\frac{12+3}{3}=5.$

Hence the numbers are 2, 5, 8, 11, 14.

GEOMETRIC PROGRESSION.

To find the common ratio of a series of Quantities in Geometric Progression.

(p. 110.)—Ex. 3. Find the *common ratio* in the series $\frac{5}{3}, 1, \frac{3}{5}, \frac{9}{25}$, &c.

Here the *common ratio* $=1\div\frac{5}{3}=1\times\frac{3}{5}=\frac{3}{5}.$

To find the sum of a Geometric series.

(p. 111.)—Ex. 3. Find the sum of 7 terms of the series, 1, 3, 9, 27, 81, &c.

$$\left.\begin{aligned}\text{Here } a&=1\\ r&=3\\ n&=7\end{aligned}\right\}\quad \therefore\ S=\frac{ar^n-a}{r-1}=\frac{1\times3^7-1}{3-1}$$

$$=\frac{3^4\times3^3-1}{2}$$

$$=\frac{81\times27-1}{2}$$

$$=\frac{2187-1}{2}=1093.$$

(p. 112.)—Ex. 4. Find the sum of 1, 2, 4, 8, 16, &c. to 14 terms.

$$\left.\begin{aligned}\text{Here } a&=1\\ r&=2\\ n&=14\end{aligned}\right\}\quad \therefore\ S=\frac{ar^n-1}{r-1}=\frac{1\times2^{14}-1}{2-1}$$

$$=4^7-1$$

$$=4^4\times4^3-1$$

$$=256\times64-1$$

$$=16384-1=16383.$$

Ex. 5. Find the sum of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c. to 8 terms.

$$\left.\begin{aligned}\text{Here } a&=1\\ r&=\frac{1}{3}\\ n&=8\end{aligned}\right\}\quad \therefore\ S=\frac{a-ar^n}{1-r}=\frac{1-1\times\left(\frac{1}{3}\right)^8}{1-\frac{1}{3}}$$

$$=\frac{\left\{1-\left(\frac{1}{3}\right)^8\right\}\times3}{3-1}$$

$$\text{Now }\left(\frac{1}{3}\right)^8=\frac{1}{3^8}=\frac{1}{6561}$$

$$\therefore\ 1-\left(\frac{1}{3}\right)^8=1-\frac{1}{6561}=\frac{6560}{6561}$$

$$\text{And } S=\frac{6560\times3}{6561\times2}=\frac{3280}{2187}.$$

(p. 112.)—Ex. 7. Find two geometric means between 4 and 256.

$$\left.\begin{aligned} \text{Here } a &= 4 \\ l &= 256 \\ n &= 4 \end{aligned}\right\} \qquad \begin{aligned} \text{and } ar^{n-1} &= l \\ \therefore\ 4r^3 &= 256, \\ r^3 &= 64, \\ \therefore\ r &= 4. \end{aligned}$$

And the required means are 16 and 64.

Ex. 8. Find three geometric means between $\frac{1}{9}$ and 9.

$$\left.\begin{aligned} \text{Here } a &= \frac{1}{9} \\ l &= 9 \\ n &= 5 \end{aligned}\right\} \qquad \begin{aligned} \therefore\ ar^{n-1} &= l \\ \frac{1}{9}r^4 &= 9, \\ r^4 &= 81; \\ \therefore\ r &= 3. \end{aligned}$$

Hence the three required means are $\frac{1}{3}$, 1, 3.

Ex. 10. What is the geometric mean between 16 and 64?

$$\left.\begin{aligned} \text{Here } a &= 16 \\ l &= 64 \\ n &= 3 \end{aligned}\right\} \qquad \begin{aligned} \text{and } ar^{n-1} &= l \\ \therefore\ 16r^2 &= 64, \\ r^2 &= 4; \\ \therefore\ r &= 2. \end{aligned}$$

Therefore the mean required is 32.

A second Solution.

Let x be the geometric mean required.

Then 16, x, 64, are three terms of a geometrical progression, and since the *common ratio* (Alg. Art. 77.) is equal to *any term divided by the one preceding it*, we have

$$\frac{x}{16} = \frac{64}{x}$$

$$\text{or } x^2 = 64 \times 16;$$

$$\therefore\ x = 8 \times 4 = 32.$$

(p. 112.)—Ex. 11. Insert four geometric means between $\frac{1}{3}$ and 81.

Here $a = \frac{1}{3}$, $l = 81$, $n = 6$; ∴ since $ar^{n-1} = l$

$$\frac{1}{3}r^5 = 81 = 3^4$$

$$r^5 = 3^5;$$

$$\therefore r = 3.$$

The means therefore are 1, 3, 9, 27.

PROBLEMS.

PROB. 2. There are three numbers in geometric progression whose *product* is 64, and *sum* 14. What are the numbers?

Let $\frac{x}{y}$, x, xy, be the numbers. Then by the problem,

$$\frac{x}{y}+x+xy = 14,$$

$$\frac{x}{y}\times x\times xy = 64.$$

These equations become by reduction

$$x\left(\frac{1}{y}+1+y\right) = 14 \quad \ldots\ldots \quad (1)$$

$$x^3 = 64, \text{ or } x = 4 \quad \ldots\ldots \quad (2)$$

Substitute the value of x in (2), in (1), and

$$4\left(\frac{1}{y}+1+y\right) = 14$$

$$\frac{1}{y}+1+y = \frac{7}{2}$$

$$y^2 - \frac{5}{2}y = -1$$

$$y^2 - \frac{5}{2}y + \left(\frac{5}{4}\right)^2 = \frac{25}{16} - 1 = \frac{9}{16}$$

$$y-\frac{5}{4}=\pm\frac{3}{4}$$

$$\therefore y=2 \text{ or } \frac{1}{2}.$$

Hence the numbers are 2, 4, 8, or 8, 4, 2.

(p. 112.)—PROB. 3. There are three numbers in geometric progression whose *sum* is 21, and the *sum of their squares* 189. What are the numbers?

Let $\frac{x}{y}$, x, xy, denote the numbers;

Then by the conditions of the problem,

$$\frac{x}{y}+x+xy=21$$

$$\frac{x^2}{y^2}+x^2+x^2y^2=189.$$

These equations may be thus exhibited:

$$x\left(\frac{1}{y}+1+y\right)=21 \quad \ldots\ldots \quad (1)$$

$$x^2\left(\frac{1}{y^2}+1+y^2\right)=189 \quad \ldots\ldots \quad (2)$$

By squaring (1), it becomes

$$x^2\left(\frac{1}{y^2}+\frac{2}{y}+3+2y+y^2\right)=441,$$

and (2) $x^2\left(\frac{1}{y^2} \quad +1 \quad +y^2\right)=189$

$\therefore$ by subtraction, $x^2\left(\frac{2}{y}+2+2y\right)=252,$

$$\text{or } x^2\left(\frac{1}{y}+1+y\right)=126.$$

Substituting (1) in this eq^n. $21x=126$;

$$\therefore x=6.$$

This value of x being put in (1)

$$6\left(\frac{1}{y}+1+y\right)=21$$

$$\frac{1}{y}+1+y=\frac{7}{2}$$

$$y^2-\frac{5}{2}y=-1$$

$$y^2-\frac{5}{2}y+\left(\frac{5}{4}\right)^2=\frac{25}{16}-1=\frac{9}{16}$$

$$y-\frac{5}{4}=\pm\frac{3}{4};$$

$$\therefore\ y=2 \text{ or } \frac{1}{2}$$

Hence the required numbers are 3, 6, 12.

This problem may also be solved in the following manner:

Let x, y, z, denote the numbers;

Then by the problem, $x+y+z=21$ ...... (1)

$x^2+y^2+z^2=189$ ...... (2)

And by geom. progression, $xz=y^2$ ...... (3)

By squaring (1) $x^2+y^2+z^2+2xy+2xz+2yz=441$

and (2) $x^2+y^2+z^2=189$

$\therefore$ by subtr. we get, $2xy+2xz+2yz=252$,

or $xy+xz+yz=126$.

But by (3) $xz=y^2$, $\therefore$ $xy+y^2+yz=126$,

or $(x+y+z)y=126$.

But by (1) $x+y+z=21$; $\therefore$ $21y=126$,

or $y=6$.

This value of y being substituted in (1) and (3), these equations become

$x+z=15$ ...... (α)

$xz=36$.

By squaring the 1st equation, and multiplying the 2d by 4, we have

$$\begin{array}{rl} x^2+2xz+z^2 &= 225 \\ 4xz &= 144 \\ \hline x^2-2xz+z^2 &= 81. \end{array}$$

Extracting the sq. root, $x-z=\pm 9$

and (α) $x+z=15$

$\therefore$ by addition, $2x=24$ or 6

or $x=12$ or 3,

and by subtraction, $2z=6$ or 24

or $z=3$ or 12.

Hence the numbers are 3, 6, 12, as before.

(p. 112.)—PROB. 4. There are three numbers in geometric progression; the sum of the *first* and *last* is 52, and the *square of the mean* is 100. What are the numbers?

Let $\frac{x}{y}$, x, xy, be the numbers;

$$\left.\begin{array}{r} \text{Then } \frac{x}{y}+xy=52, \text{ or } x\left(\frac{1}{y}+y\right)=52 \\ \text{and } x^2=100, \text{ or } x=10 \end{array}\right\}$$

Therefore by dividing the 1st of these equations by the 2d, we get

$$\frac{1}{y}+y=\frac{26}{5}$$

$$y^2-\frac{26}{5}y=-1$$

$$y^2-\frac{26}{5}y+\left(\frac{13}{5}\right)^2=\frac{169}{25}-1=\frac{144}{25}$$

$$y-\frac{13}{5}=\pm\frac{12}{5};$$

$$\therefore y=5 \text{ or } \frac{1}{5}.$$

Hence the numbers are 2, 10, 50.

(p. 112.)—PROB. 5. There are three numbers in geometric progression, whose sum is 31, and the sum of the *first* and *last* is 26. What are the numbers?

Let $\frac{x}{y}$, x, xy, represent the numbers;

$$\text{Then } \frac{x}{y}+x+xy=31 \quad \ldots\ldots \quad (1)$$

$$\frac{x}{y}+xy=26 \quad \ldots\ldots \quad (2)$$

By subtr^g. (2) from (1), $x=5$.

And by substituting this value of x in (2), we get the equation

$$\frac{1}{y}+y=\frac{26}{5}$$

$$y^2-\frac{26}{5}y=-1$$

$$y^2-\frac{26}{5}y+\left(\frac{13}{5}\right)^2=\frac{169}{25}-1=\frac{144}{25}$$

$$y-\frac{13}{5}=\pm\frac{12}{5};$$

$$\therefore y=5 \text{ or } \frac{1}{5}.$$

Therefore the required numbers are 1, 5, 25.

ON THE SUMMATION OF AN INFINITE SERIES OF FRACTIONS IN GEOMETRIC PROGRESSION.

(p. 114.)—Ex. 3. Find the value of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$ &c. *ad infinitum.*

$$\left.\begin{aligned}\text{Here } a&=1\\ r&=\frac{1}{3}\end{aligned}\right\} \quad \therefore S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{3}}=\frac{3}{3-1}=\frac{3}{2}$$

(p. 114.)—Ex. 4. Find the value of $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+$ &c. *ad infinitum.*

$$\left.\begin{array}{r}\text{Here } a=1\\ r=\frac{3}{4}\end{array}\right\} \therefore S=\frac{1}{1-\frac{3}{4}}=\frac{4}{4-3}=4.$$

Ex. 5. Find the value of $\frac{2}{5}+\frac{4}{25}+\frac{8}{125}+$ &c. *ad infinitum.*

$$\left.\begin{array}{r}\text{Here } a=\frac{2}{5}\\ r=\frac{2}{5}\end{array}\right\} \therefore S=\frac{\frac{2}{5}}{1-\frac{2}{5}}=\frac{\frac{2}{5}}{\frac{3}{5}}=\frac{2}{3}$$

To find the value of Circulating Decimals.

(p. 116.)—Ex. 5. Find the value of .77777 &c. *ad infinitum.*

$$\text{Since } \cdot77777 \ldots =\frac{7}{10}+\frac{7}{100}+\frac{7}{1000}+\frac{7}{10000}+\ldots\ldots$$

$$\left.\begin{array}{r}a=\frac{7}{10}\\ r=\frac{1}{10}\end{array}\right\} \therefore S=\frac{a}{1-r}=\frac{\frac{7}{10}}{1-\frac{1}{10}}=\frac{\frac{7}{10}}{\frac{9}{10}}=\frac{7}{9}$$

Ex. 6. Find the value of .232323 &c. *ad infinitum.*

$$.232323 \ldots =\frac{23}{100}+\frac{23}{10000}+\frac{23}{1000000}+\ldots\ldots$$

$$\left.\begin{array}{r}a=\frac{23}{100}\\ r=\frac{1}{100}\end{array}\right\} \therefore S=\frac{\frac{23}{100}}{1-\frac{1}{100}}=\frac{\frac{23}{100}}{\frac{10\ -1}{100}}=\frac{23}{99}.$$

(p. 116.)—Ex. 7. Find the value of .83333 &c. *ad infinitum.*

Since $.8333 \ldots\ldots = \frac{8}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \ldots\ldots$

$$\left.\begin{array}{l} a = \frac{3}{100} \\ r = \frac{1}{10} \end{array}\right\} \therefore S = \frac{a}{1-r} = \frac{\frac{3}{100}}{1-\frac{1}{10}} = \frac{\frac{3}{100}}{\frac{9}{10}} = \frac{1}{30},$$

and the value of the decimal $= \frac{8}{10} + \frac{1}{30} = \frac{25}{30} = \frac{5}{6}$.

Ex. 8. Find the value of .7141414 &c. *ad infinitum.*

$$\left.\begin{array}{l} \text{Here } a = \frac{14}{1000} \\ r = \frac{1}{100} \end{array}\right\} \therefore S = \frac{\frac{14}{1000}}{1-\frac{1}{100}} = \frac{14}{1000-10} = \frac{14}{990},$$

and the value of the decimal $= \frac{7}{10} + \frac{14}{990} = \frac{707}{990}$.

Ex. 9. Find the value of .956666 &c. *ad infinitum.*

$$\left.\begin{array}{l} a = \frac{6}{1000} \\ r = \frac{1}{10} \end{array}\right\} \therefore S = \frac{a}{1-r} = \frac{\frac{6}{1000}}{1-\frac{1}{10}} = \frac{6}{1000-100} = \frac{2}{300};$$

$\therefore$ the value of the decimal $= \frac{95}{100} + \frac{2}{300} = \frac{285}{300} + \frac{2}{300} = \frac{287}{300}$.

The *five* preceding examples on Circulating Decimals can be solved more simply, by the *second method* given in the Algebra, as follows:

(p. 116.)—Ex. 5.

$$\begin{aligned} \text{Let } S &= .7777 \ldots. \\ \text{then } 10\,S &= 7.7777 \ldots. \\ \hline \therefore\ 9\,S &= 7 \\ \text{and } S &= \frac{7}{9} \end{aligned}$$

(p. 116)—Ex. 6.

$$\text{Let } S = .232323 \ldots$$
$$100\,S = 23.232323 \ldots$$
$$\therefore\ 99\,S = 23;$$
$$\text{and } \therefore\ S = \frac{23}{99}.$$

Ex. 7.

$$\text{Let } S = .8333 \ldots$$
$$\text{then } 100\,S = 83.333 \ldots$$
$$\text{and } 10\,S = 8.333 \ldots$$
$$\therefore\ 90\,S = 75;$$
$$\therefore\ S = \frac{75}{90} = \frac{5}{6}.$$

Ex. 8.

$$\text{Let } S = .7141414 \ldots$$
$$1000\,S = 714.1414 \ldots$$
$$10\,S = 7.1414 \ldots$$
$$\therefore\ 990\,S = 707;$$
$$\text{and } \therefore\ S = \frac{707}{990}.$$

Ex. 9.

$$\text{Let } S = \cdot 956666 \ldots$$
$$1000\,S = 956.666 \ldots$$
$$100\,S = 95.666 \ldots$$
$$\therefore\ 900\,S = 861;$$
$$\text{and } \therefore\ S = \frac{861}{900} = \frac{287}{300}.$$

(p. 116.)—Prob. 1. A body in motion moves over 1 mile in the *first* second, but being acted upon by some retarding cause, it only moves over $\frac{1}{2}$ a mile in the *second* second, $\frac{1}{4}$ the *third*, and so on. Show that, according to this law of motion, the body, though it move on to *all eternity*, will never pass over a space greater than 2 miles.

The spaces passed over by the body in *successive* seconds are 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{10}$, &c., *ad infinitum*, miles: we have therefore, in solving this problem, only to find the *sum* of an infinite geometric series, in which

$$\left.\begin{array}{l} a=1 \\ r=\frac{1}{2} \end{array}\right\} \text{ and } \therefore S=\frac{a}{1-r}=\frac{1}{1-\frac{1}{2}}=\frac{2}{2-1}=2 \text{ miles,}$$

a distance beyond which the body can never travel.

HARMONIC PROGRESSION.

(p. 118.)—Ex. 4. Find a harmonic mean between 12 and 6.

Let x be the mean required ;

Then $\frac{1}{12}$, $\frac{1}{x}$, $\frac{1}{6}$, are in arithmetic progression ;

$$\therefore \frac{1}{12}-\frac{1}{x}=\frac{1}{x}-\frac{1}{6}$$

$$\frac{2}{x}=\frac{1}{12}+\frac{1}{6}$$

$$=\frac{1}{4};$$

$$\therefore x=8.$$

Ex. 5. The numbers 4 and 6 are two terms of a harmonic progression ; find a third term.

Let x be the third term required ;

Then $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{x}$, are terms in arithmetic progression ;

$$\therefore \frac{1}{4}-\frac{1}{6}=\frac{1}{6}-\frac{1}{x}$$

$$\frac{1}{x}=\frac{2}{6}-\frac{1}{4}$$

$$=\frac{1}{12}$$

$$\therefore x=12.$$

(p. 118.)—Ex. 6. Find two harmonic means between 84 and 56.

Here $a=\frac{1}{84}$, $l=\frac{1}{56}$, $n=4$

$$a+(n-1)d=l$$

$$\therefore \frac{1}{84}+(4-1)d=\frac{1}{56}$$

$$3d=\frac{1}{56}-\frac{1}{84}$$

$$=\frac{84-56}{56\times 84}$$

$$=\frac{28}{56\times 84}$$

$$=\frac{1}{168};$$

$$\therefore d=\frac{1}{504}:$$

Hence $a+d=\frac{1}{84}+\frac{1}{504}=\frac{7}{504}=\frac{1}{72}$

$$a+2d=\frac{1}{84}+\frac{2}{504}=\frac{8}{504}=\frac{1}{63};$$

and therefore the required means are 72 and 63.

E 7. Insert three harmonic means between 15 and 3.

Here $a=\frac{1}{15}$, $l=\frac{1}{3}$, $n=5$

And $a+(n-1)d=l$

$$\therefore \frac{1}{15}+(5-1)d=\frac{1}{3}$$

$$4d=\frac{4}{15}$$

$$\therefore d=\frac{1}{15}:$$

$$\text{And } a+d=\frac{1}{15}+\frac{1}{15}=\frac{2}{15}$$

$$a+2d=\frac{1}{15}+\frac{2}{15}=\frac{1}{5}$$

$$a+3d=\frac{1}{15}+\frac{3}{15}=\frac{4}{15}:$$

Hence the required means are $\frac{15}{2}$, 5, $\frac{15}{4}$.

PERMUTATIONS.

(p. 123.)—Ex. 4. How many permutations can be formed out of 10 letters, taken 5 at a time?

$$\left.\begin{array}{l} n=10 \\ r=\ \ 5 \end{array}\right\} \therefore \text{the No.}=n(n-1)(n-2)\ldots(n-r+1)$$

$$=10(10-1).(10-2)(10-3)(10-4)$$

$$=10.9.8.7.6$$

$$=30240.$$

Ex. 5. How many permutations can be formed out of the words *Algebra* and *Missippi* respectively, all the letters being taken at once?

(1.) In the word *Algebra*, there are 7 letters, and a occurs *twice;*

$$\therefore \text{the No. of permutations}=\frac{7.6.5.4.3.2.1}{1.2}=2520.$$

(2.) In *Missippi* there are 8 letters, and i occurs *thrice, s twice*, and p *twice;*

$$\therefore \text{the No. of permutations}=\frac{8.7.6.5.4.3.2.1}{1.2.3\times1.2\times1.2}=1680.$$

COMBINATIONS.

(p. 124.)—Ex. 3. Find how many different combinations of 8 letters, taken in every possible way, can be made.

The general expression for the *number* of combinations of n things taken r together is $\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}$;

and therefore in this example,

when $r=1$, $\frac{n}{1}$ .... $=\frac{8}{1}$ .... $= 8$

...... $r=2$, $\frac{n(n-1)}{1.2}$ .... $=\frac{8.7}{1.2}$ .... $= 28$

...... $r=3$, $\frac{n(n-1)(n-2)}{1.2.3}$ ...... $=\frac{8.7.6}{1.2.3}$ .... $= 56$

...... $r=4$, $\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}=\frac{8.7.6.5}{1.2.3.4}$ .... $= 70$

...... $r=5$, $\frac{n(n-1)\dots(n-4)}{1.2\dots\dots\dots 5}$ ... $=\frac{8.7.6.5.4}{1.2.3.4.5}$ .... $= 56$

...... $r=6$, $\frac{n(n-1)\dots(n-5)}{1.2\dots 6}$ ... $=\frac{8.7.6.5.4.3}{1.2.3.4.5.6}$ ...... $= 28$

...... $r=7$, $\frac{n(n-1)\dots(n-6)}{1.2\dots\dots 7}$ ... $=\frac{8.7.6.5.4.3.2}{1.2.3.4.5.6.7}$ ... $= 8$

...... $r=8$, $\frac{n(n-1)\dots(n-7)}{1.2\dots 8}$ ... $=\frac{8.7.6.5.4.3.2.1}{1.2.3.4.5.6.7.8}$ $\therefore = 1$

Hence the total number of combinations $= 255$

It may be observed that this solution, which is deduced from the general formula, is more comprehensive than the *definition* (Alg. Art. 89), which does not include the case when $r = 1$, appears to warrant. The total number of combinations of *six* colours (Ex. 2.) have in like manner been found to be 63 instead of 57. The language of Algebra from its generality is indeed far more extensive in its signification than that of ordinary discourse; and for this reason, it is of the utmost importance to be able to *interpret* correctly—especially in the application of the algebraic analysis to researches in physical science, one of the most important objects of mathematical study—the results which may arise in the solution of a problem.

THE END.

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